A mathematical derivation of the Maxwell equations

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Abstract: Waves of all types are described mathematically using partial differential equations. Here, departing from this tradition, I describe waves using a novel system of three simultaneous vector algebraic equations. These equations when set in the electromagnetic domain are a novel mathematical reformulation of the Maxwell equations: $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E}) = \left\{\mathbf{E} = \mathbf{u}\times\mathbf{B};\ \mathbf{u} = (\mathbf{B}\times\mathbf{E})/\|\mathbf{B}\|^2;\ \mathbf{B} = (\mathbf{E}\times\mathbf{u})/\|\mathbf{u}\|^2\right\}$ where $\mathbf{u}$ is a velocity vector. Furthermore, the expressions for the permittivity ϵ_0, permeability μ_0 and the magnetic flux density $\mathbf{B}$ are obtained by manipulating $\mathscr{M}$. As an application of $\mathscr{M}$ I show that three dimensional spherical EM-wave structures do exist, in theory at least. They are stationary with finite dimensionality and could provide the basis for describing EM-solitons, which in turn could be used to describe many natural phenomena, including ball lightning among others.

Keywords: Wave equation, Maxwell equations, EM-waves, EM-soliton, Ball lightning, Bimodal waves

What is a wave? Towne [1] states that the requirement for a physical condition to be referred to as a wave, is that its mathematical representation give rise to a partial differential equation of particular form, known as the wave equation. The classical form

$$\frac{\partial^2 w}{\partial p^2} - \frac{1}{u^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{or} \qquad \nabla^2 w - \frac{1}{u^2}\frac{\partial^2 w}{\partial t^2} = 0.$$

was proposed in 1748 by d'Alembert for a one-dimensional continuum. A decade later, Euler established the equation for the three-dimensional continuum. The above wave equations describe a disturbance with motion but do not indicate, in the first place, why the wave is possible at all. A physical wave is a state alternating between two domains; only one of the domains is represented in the equations above.

Here we do not concern ourself with the classical, or the d'Alembert's, wave equation. Instead I present a novel wave equation system consisting of three simultaneous vector equations. These were discovered by the fortuitous penning of the following sequence:

$$\mathbf{r}_1 = \mathbf{u}_0 \times \mathbf{a}_0, \quad \mathbf{u}_1 = \mathbf{a}_0 \times \mathbf{r}_1, \quad \mathbf{a}_1 = \mathbf{r}_1 \times \mathbf{u}_1, \quad \mathbf{r}_2 = \mathbf{u}_1 \times \mathbf{a}_1, \quad \cdots$$

and I soon realised that the sequence can continue unaltered indefinitely; that is $\mathbf{u}_n = \mathbf{u}$, $\mathbf{a}_n = \mathbf{a}$ and $\mathbf{r}_n = \mathbf{r}$, but only if normalisation is introduced:

$$\mathbf{r} = \mathbf{u} \times \mathbf{a}, \quad \mathbf{u} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a} \times \mathbf{r}, \quad \mathbf{a} = \frac{1}{\|\mathbf{u}\|^2}\mathbf{r} \times \mathbf{u}$$

Now, if the vectors are also functions of time, a new wave equation-system is born, provided that **u** is a velocity vector, and both **a** and **r** are vector domains that complement each other to facilitate the wave action. The solution of the above set of three simultaneous vector equations describes bimodal-transverse waves.

As a general note, throughout this article, when referring to the Maxwell equation, I refer to the Maxwell equations in vacuum:

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

1 The reformulated Maxwell equations

In an Euclidean $\mathbb{R}^3$ homogeneous space xyz, where $\hat{z} = \hat{x} \times \hat{y}$, we consider one plane $\mathcal{W}(\mathbf{p})$ of an EM-travelling plane wave and where **p** defines the position of $\mathcal{W}$. Such a wave is described (mathematically expressed: $\xrightarrow[\text{by}]{\text{dsc}}$) by the solution of $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$, which is a system of three simultaneous vector algebraic equations:

$$\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E}) = \left\{ \begin{array}{lll} \mathbf{E} = \mathbf{u} \times \mathbf{B} & (\textit{activation by } \mathbf{B}) & \text{(a)} \\ \mathbf{u} = \dfrac{1}{\|\mathbf{B}\|^2}\mathbf{B} \times \mathbf{E} & (\textit{vectoring by } \mathbf{B} \times \mathbf{E}) & \text{(b)} \\ \mathbf{B} = \dfrac{1}{\|\mathbf{u}\|^2}\mathbf{E} \times \mathbf{u} & (\textit{reactivation by } \mathbf{E}) & \text{(c)} \end{array} \right\} \tag{1}$$

Here the terms activation and re-activation are synonymous with self-induction. The above equation set is a reformulation of the Maxwell equations when:

u is a velocity vector $\mathbf{u} = c\hat{\mathrm{u}}$, where
$\hat{\mathrm{u}}$ that is, $\hat{\mathrm{u}} \mapsto \hat{\mathrm{u}}(t)$, is a unitless unit vector function of time only, and
c is the speed of light.

B is the magnetic field $\mathbf{B} = B\hat{\mathrm{B}}$, and where

$\hat{\mathrm{B}}$ that is, $\hat{\mathrm{B}} \mapsto \hat{\mathrm{B}}(t)$, is a unitless unit vector function of time only, and is orthogonal to $\hat{\mathrm{u}}$ hence $\hat{\mathrm{u}} \cdot \hat{\mathrm{B}} = 0$, and

B scales the magnetic field and provides the physical units.

E is the electric field and (1)(a) gives $\mathbf{E} = cB(\hat{\mathrm{u}} \times \hat{\mathrm{B}}) = cB\hat{\mathrm{E}}$, with $\hat{\mathrm{E}} = \hat{\mathrm{u}} \times \hat{\mathrm{B}}$.

p the position of the origin for **u**, **B**, and **E**; thus $\mathbf{p} = \int \mathbf{u}\,\mathrm{d}t$.

That (1) is a mathematical reformulation of the Maxwell equations is demonstrated as follows: First we need to evaluate the triple vector products $\nabla \times (\mathbf{u} \times \mathbf{B})$ and $\nabla \times (\mathbf{E} \times \mathbf{u})$, which we expand using general vector analytic methods.

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{u}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{u}) + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}$$

$$\nabla \times (\mathbf{E} \times \mathbf{u}) = \mathbf{E}(\nabla \cdot \mathbf{u}) - \mathbf{u}(\nabla \cdot \mathbf{E}) + (\mathbf{u} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{u}$$

Evaluating the terms

$$\nabla \cdot \mathbf{u} = 0 \quad \text{because } c \text{ and } \hat{\mathrm{u}}(t) \text{ are not functions of } x,\ y, \text{ and } z$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{because } B \text{ and } \hat{\mathrm{B}}(t) \text{ are not functions of } x,\ y, \text{ and } z \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0 \quad \text{ditto, because } \mathbf{E} = \mathbf{u} \times \mathbf{B} \tag{3}$$

$$(\mathbf{B} \cdot \nabla)\mathbf{u} = 0 \quad \text{because } \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right) c\hat{\mathrm{u}}(t) = 0$$

$$(\mathbf{E} \cdot \nabla)\mathbf{u} = 0 \quad \text{ditto}$$

$$\mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} \quad \text{because } \mathbf{u} \cdot \nabla = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial t}$$

and that leaves us with

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times (\mathbf{E} \times \mathbf{u}) = \frac{\partial \mathbf{E}}{\partial t}$$

Applying a 'left and right side' curl operation on (1)(a) and (c) and using the above we are well on the way to recover the Maxwell equations in vacuum with (2) through (5) below.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{5}$$

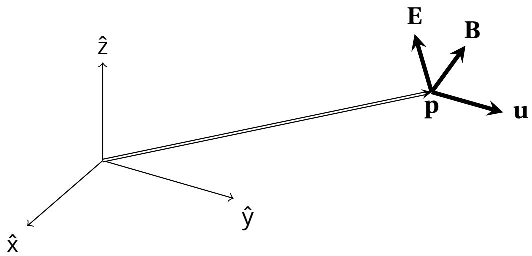

Figure 1: Illustrating the vectors used in $\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$.

and it is well known that a further manipulation of the equations (2–5) gives the wave equations

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \text{and} \quad \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

proving that $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ is a new formulation for bimodal-waves such as EM-waves.

To prove that $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ is also a reformulation of the Maxwell equations, we can take the easy path and simply substitute $c^2 = 1/\epsilon_0\mu_0$ in the above. The more difficult path is to assert

a) An elementary EM-wave $\mathcal{W}$ exhibits power h/t^2, where h is the Planck constant and $t = 1$ second. This requires $\mathbf{B}$ to be an elementary field.

b) This elementary wave transports an electric charge e every one second which is a *wave current*. (This is nothing new; it is another way of describing the displacement current $\partial\mathbf{E}/\partial t$ that Maxwell had identified in varying electric fields.)

and then show that (1) together with the above assertions demands ϵ_0 and μ_0 in the form that they are known to us.

From the definitions of (1) we have $\|\mathbf{B}\| = |\mathbf{B}| = B$ which we substitute into (1)(b) to obtain

$$\mathbf{u} = \frac{1}{B^2}\mathbf{B} \times \mathbf{E} \tag{6}$$

On the premise that $\mathbf{B} \times \mathbf{E}$ is indicative of the wave action, we multiply (6) by the quantised action h and evaluate the norms

$$\|h\mathbf{u}\| = \left\| \frac{h}{B^2}\mathbf{B}\times\mathbf{E} \right\|$$

$$\therefore \qquad hc = \left(\frac{h}{B^2}\right)\left(|\mathbf{B}||\mathbf{E}|\right)$$

We define an elementary distance $l = ct$, and multiplying and dividing the above by l^4 to transform the above from a domain of fields to a domain of fluxes, gives

$$h = \left[\frac{h}{l^4 B^2 c}\right] l^4\left(|\mathbf{B}||\mathbf{E}|\right) \tag{7}$$

but that also requires $\mathbf{B}$ and $\mathbf{E}$ to be elementary fields. Here the square brackets indicate the development of a physical constant, which we want to determine by eliminating B.

Let's define the elementary electromagnetic action as $h_e = \varrho h$ where $\varrho = 1$ C kg^{-1} a correction factor to satisfy the dimensionality when working with electromagnetic quantities. Action is momentum times distance. Using a mechanical analogy we can say that EM-momentum is proportional to electric charge times velocity, having units of coulomb metres per second. As the wave transports an elementary charge e (Assertion-2 above) at a velocity c, the EM-wave action is EM-momentum times distance; here we consider the elementary distance $l = ct$ measured longitudinally to the propagation direction. Therefore, the EM-wave action is also

$$h_e = \varrho h = \kappa lec \tag{8}$$

and where κ is a dimensionless proportionality constant of unknown value, scaling lec to the EM-action h_e.

Let us think about $\mathbf{B}$ in the context of the EM-wave and the above assertions; it facilitates the transportation of the charge e. Because the charge is carried by an EM-wave we can also postulate that the electromagnetic action is proportional to the product of $\mathbf{B}$ and the elementary volume which the wave occupies

$$\varrho h = \chi l^3|\mathbf{B}|$$

and where χ is a constant with units and scaling to be determined. Combining the above with (8) gives

$$|\mathbf{B}| = \frac{\kappa ec}{\chi l^2}$$

and we substitute $|\mathbf{B}|$ from the above into (7) to get

$$h = \left[\frac{h}{l^4 B^2 c}\right] l^4 \Big(\frac{\kappa e c}{\chi l^2} |\mathbf{E}|\Big)$$

but, $|\mathbf{E}| = cB$ which gives

$$h = \left[\frac{h}{l^4 B^2 c}\right] \left[\frac{1}{\chi}\right] \kappa l^2 e c^2 B$$

We are now in the position to define the expression for

$$B = \frac{h}{\kappa l^2 e} \tag{9}$$

but only if

$$1 = \left[\frac{h}{l^4 B^2 c}\right] \left[\frac{1}{\chi}\right] c^2 \qquad \text{and replacing } B \text{ using (9) gives}$$

$$1 = \left[\frac{\kappa^2 e^2}{hc}\right] \left[\frac{1}{\chi}\right] c^2 \qquad \text{which requires } \frac{1}{\chi} = \frac{h}{\kappa^2 e^2 c}, \text{ hence}$$

$$1 = \left[\frac{\kappa^2 e^2}{hc}\right] \left[\frac{h}{\kappa^2 e^2 c}\right] c^2 \tag{10}$$

Now—with a bit of hindsight—all that remains is to set

$$\kappa^2 = \frac{1}{2\alpha}$$

where α is the fine structure constant. Equation (10) now gives the sought after result

$$\epsilon_0 = \frac{e^2}{2\alpha hc} \quad \text{and} \quad \mu_0 = \frac{2\alpha h}{e^2 c} \tag{11}$$

expressions first formulated in 1916 by Sommerfeld [2] but in a way to define the fine structure constant α.

This concludes the proof that the equation set (1); that is $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$, is a mathematical reformulation of the Maxwell equations, because now we can replace $1/c^2$ in (5) with $\epsilon_0 \mu_0$ as we have derived it independently.

In all of the above I have not resorted to any electromagnetic theories. Admittedly I have fine tuned the last step to obtain results in accord with the accepted definitions of the physical constants, but that should not deter us; it is what experimental physicists do daily, that is, determining constants from experimental results to match theory.

Now, it is beyond any doubt that the equation set (1) together with the two assertions are fundamental to Nature. How else does one explain that the correct expressions for the electromagnetic quantities are obtained on a mathematical basis without resort to an expanse of experimental observations. This validates the raising of numerous points and questions:

- Analytical simplicity predicts new EM-waveforms

Electromagnetic waves as described by the Maxwell equations in free space are well studied and well understood, but only in the singular context as solutions to the d'Alembert wave equation. That means that for any solution the magnetic and electric field are always expressed as a function of both position and time. Researchers then make use of the superposition to construct intricate wave structures for their analysis.

The equation set $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ provides not only a new understanding of the underlying requirements for EM-waves; it also predicts new EM-wave types because the three vectors **u**, **B**, and **E** are defined in time only and the limitations dictated by solving the d'Alembert wave equation fall away.

- From the fine structure constant to elementary time and distance

The fine structure constant α is said to quantify the strength of the electromagnetic interaction between elementary charged particles; the modern view also includes the coupling of the electromagnetic force to the other three forces [3]; these are the strong, weak and gravitational forces. Repeating (8), $\varrho h = \kappa lec$, here the constant $\kappa = 1/\sqrt{2\alpha}$ is a coupling constant relating the electric charge momentum to mechanical momentum.

Oliver Heaviside [4] in 1892 presented us with vector algebra; he used it to recast Maxwell's original 20 equations to the four equations that we now recognise as the Maxwell equations. Now let's suppose the events of history were different; it is conceivable that Heaviside could also have stumbled on the equation set (1). In the year 1900 Planck proposed the quantity h and the value for the electric charge e was also known; so under the above supposition it is very probable that a Heaviside constant $\kappa = 8.277$ would have been proposed and the magnetic permeability would have transitioned from the fixed constant $4\pi \times 10^{-7}$ to the expression $h/(\kappa^2 e^2 c)$, or at least that relationship would have been known. Then sixteen years later, Sommerfeld would have established $\alpha^{-1} = 2\kappa^2$.

Nonetheless, $\varrho h = \kappa lec$ now provides a key to determine the values for the elementary length and time. Using the 2018 CODATA values we get:

$$\begin{aligned}
\kappa &= 8.277\,559\,999\,29(62) && \text{Heaviside constant} \\
l_0 &= 1.666\,566\,299\,11(12)\times 10^{-24} && \text{metres} \\
t_0 &= 5.559\,066\,796\,49(42)\times 10^{-33} && \text{seconds} \qquad \text{using } l = ct.
\end{aligned}$$

- Poynting vector: Not $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ but $\mathbf{S} = \mathbf{H}\times\mathbf{E}$.

  The Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ and the associated electromagnetic momentum $\mathbf{g} = \mathbf{E}\times\mathbf{B}$ origins are in (1)(b), that is $\mathbf{u} = \|\mathbf{B}\|^{-2}\mathbf{B}\times\mathbf{E}$, but are of opposite sign by reason of prior choices and conventions. Recapitulating Jackson [5]: "*The wave equations*

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad \textit{and} \quad \nabla^2\mathbf{B} - \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} = 0$$

are both solved by assuming plane wave fields

$$\mathbf{E}(\mathbf{x},t) = \mathcal{E}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t} \quad \textit{and} \quad \mathbf{B}(\mathbf{x},t) = \mathcal{B}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}$$

where $\mathcal{E}$, $\mathcal{B}$ *and* $\mathbf{n}$ *are vectors that are constant in time and space. Each component of* $\mathbf{E}$ *and* $\mathbf{B}$ *satisfies the wave equations provided that* $k^2\mathbf{n}\cdot\mathbf{n} = \mu\epsilon\omega^2c^{-2}$ *and* $\mathbf{n}\cdot\mathbf{n} = 1$. *The divergence equations* $\nabla\cdot\mathbf{E} = 0$ *and* $\nabla\cdot\mathbf{B} = 0$ *demand that* $\mathbf{n}\cdot\mathcal{E} = 0$ *and* $\mathbf{n}\cdot\mathcal{B} = 0$*; this establishes the co-orthogonality of* $\mathcal{E}$, $\mathcal{B}$ *and* $\mathbf{n}$*. The curl equation* $\nabla\times\mathbf{E} = \partial\mathbf{B}/\partial t$ *demands a further restriction*

$$\mathcal{B} = \sqrt{\mu\epsilon}\,\mathbf{n}\times\mathcal{E} \tag{12}$$

implying that $\mathcal{E}$ *and* $\mathcal{B}$ *have the same phase*"—and the cross product defines the spacial orientations of $\mathcal{E}$, $\mathcal{B}$ and $\mathbf{n}$. But equally, one can choose the plane wave fields

$$\mathbf{E}(\mathbf{x},t) = \mathcal{E}e^{ik'\mathbf{n}'\cdot\mathbf{x}-i\omega t} \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) = \mathcal{B}e^{ik'\mathbf{n}'\cdot\mathbf{x}-i\omega t}$$

where $k' = -k$ and $\mathbf{n}' = -\mathbf{n}$; now $\nabla\times\mathbf{E} = \partial\mathbf{B}/\partial t$ demands the restriction

$$\mathcal{B} = \sqrt{\mu\epsilon}\,\mathbf{n}'\times\mathcal{E} \;=\; -\sqrt{\mu\epsilon}\,\mathbf{n}\times\mathcal{E} \;=\; \sqrt{\mu\epsilon}\,\mathcal{E}\times\mathbf{n} \tag{13}$$

which contradicts (12). However, it orientates the vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{u} = c\mathbf{n}$ in accord with (1)(b), and $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ demands $\mathbf{S} = \mathbf{H}\times\mathbf{E}$.

- Magnetic field quanta.

  The magnetic flux quantum is defined as $\phi_0 = h/(2e)$; it was observed experimentally in 1961 by Deaver [6] in hollow superconducting cylinders, and Shankar [7] shows how to derive it by analysing the Aharonov–Bohm effect. With (9) I found the magnetic field of an elementary EM-wave $\mathcal{W}$ as $B = h/(\kappa l^2 e)$ which implies a magnetic flux for the elementary EM-wave $\tilde{\phi} = h/(\kappa e) = \sqrt{2\alpha}\, h/e$ which is clearly smaller than the magnetic flux quantum established by measurement and quantum theory.

At this point I offer no opinion regarding whether $\tilde{\phi} = \sqrt{2\alpha}\, h/e$ is a quantum or not, other than to comment that when scaled this way we have the desirable result that

$$\mathbf{S} = \mu_0^{-1} \|\mathbf{B} \times \mathbf{E}\| = h \frac{c^2}{l^4} \qquad \text{energy per (area} \times \text{time)}$$

which confirms the first of the assertions on page-4.

- What defines the speed of light?

The above analysis also raises a '*Who was first? Chicken or egg*' situation with respect to the speed of light. The permittivity ϵ_0 and permeability μ_0 were derived using the velocity c defined previously in (1). The question is: *Does $c = 1/\sqrt{\epsilon_0 \mu_0}$ define the speed of light from first principles or is there another fundamental explanation to explain the velocity c?*

For example, the speed of a sound wave in a material is dependent on the material properties. In fluids $c^2 = K_s/\rho$ where K_s is a coefficient of stiffness and ρ the fluid's density. Alternatively, we can also express it as $c^2 = \partial P/\partial \rho$ where P is pressure. But do take note of the fact that none of K_s, ρ and P are defined in terms of the speed of sound within that medium.

The inference of the above is that I consider ϵ_0 and μ_0 as derived quantities. Therefore, it can be presumed that space has additional characteristics from which the *transportivity* $\mathcal{T} = c^2$ is defined. As an analog to fluids, the transportivity could be a ratio of two properties, which are not functions of the speed of light.

- New EM-wave forms

Reformulating the Maxwell equations as a wave equation $\mathcal{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ provides new mathematical descriptions for EM-waveforms not thought possible before as natural EM-phenomena. One such form is a three dimensional and "stationary" EM-wave, which periodically traverses a closed and curved, or wound up, path; possibly such waves provide a theoretical basis to explain ball lightning as an EM-soliton.

2 Describing EM-waves

To fully describe a wave $\mathcal{W}$ also requires a set of parameter equations $\mathscr{P}(\mathbf{u},\mathbf{B},\mathbf{E})$ which provide the solution to $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$. The parameter equations $\mathscr{P}(\mathbf{u},\mathbf{B},\mathbf{E})$ define the unit vectors $\hat{\mathrm{u}}(t)$, $\hat{\mathrm{B}}(t)$ and $\hat{\mathrm{E}}(t)$ all as functions of t only, with the quantifiers c, B and E providing the necessary units, or quantities, and the scaling for $\mathcal{W}$. Any set of unit vectors $\hat{\mathrm{u}}(t)$, $\hat{\mathrm{B}}(t)$ and $\hat{\mathrm{E}}(t)$ that simultaneously satisfy

$$\hat{\mathrm{E}} = \hat{\mathrm{u}} \times \hat{\mathrm{B}} \qquad \hat{\mathrm{u}} = \hat{\mathrm{B}} \times \hat{\mathrm{E}} \qquad \hat{\mathrm{B}} = \hat{\mathrm{E}} \times \hat{\mathrm{u}}.$$

provide a solution to $\mathscr{M}$. Suitable solutions can be found, among others, by a succession of Euler rotations. Now, with a solution $\mathscr{P}$ for the equation-set $\mathscr{M}$, the wave $\mathcal{W}$ is described by ($\xrightarrow[\text{by}]{\text{dsc}}$) $\mathscr{M}$ which is parameterised by ($\xrightarrow[\text{by}]{\text{par}}$) $\mathscr{P}$ and expressed as $\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E}) \xrightarrow[\text{by}]{\text{par}} \mathscr{P}(\mathbf{u},\mathbf{B},\mathbf{E})$, which we simply shorten to $\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{par}} \mathscr{P}(\mathbf{u},\mathbf{B},\mathbf{E})$ thus implicitly implying $\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ and $\mathbf{p} = \int \mathbf{u}\,\mathrm{d}t$.

2.1 Travelling plane waves

Every authoritative book, for example Jackson[5], describes an EM-field of a circular polarised travelling plane wave as

$$\begin{aligned}\mathbf{B}(\mathbf{z},t) &= B_0\big(\hat{\mathrm{x}}\cos(k\mathbf{n}\cdot\mathbf{z}-\omega t)+\hat{\mathrm{y}}\sin(k\mathbf{n}\cdot\mathbf{z}-\omega t)\big)\\ &= B_0\left[\hat{\mathrm{x}}\cos\!\left(\frac{\omega z}{c}-\omega t\right)+\hat{\mathrm{y}}\sin\!\left(\frac{\omega z}{c}-\omega t\right)\right]\end{aligned}$$

with the phase velocity defined by the wave vector $k\mathbf{n}$.

The equation set $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ defines the wave's velocity vector $\mathbf{u} = \hat{\mathrm{z}}c$ outside the field definitions. Let us introduce a position vector

$$\mathbf{p}_i = \int \mathbf{u}\,\mathrm{d}t = \hat{\mathrm{z}}\int c\,\mathrm{d}t = \hat{\mathrm{z}}(z_i + ct)$$

and use it to describe a circular polarised travelling plane EM-wave $\mathcal{W}$ as follows:

$$\mathcal{W}(\mathbf{p}_i,t) \xrightarrow[\text{by}]{\text{par}} \mathscr{P}_{\mathcal{W}}(\mathbf{u},\mathbf{B},\mathbf{E}) = \left\{\begin{aligned}&\mathbf{u} = \hat{\mathrm{z}}c\\ &\mathbf{B} = B\left[\hat{\mathrm{x}}\cos\omega\!\left(\frac{\mathbf{p}_i}{c}-t\right)+\hat{\mathrm{y}}\sin\omega\!\left(\frac{\mathbf{p}_i}{c}-t\right)\right]\\ &\mathbf{E} = cB\left[-\hat{\mathrm{x}}\sin\omega\!\left(\frac{\mathbf{p}_i}{c}-t\right)+\hat{\mathrm{y}}\cos\omega\!\left(\frac{\mathbf{p}_i}{c}-t\right)\right]\end{aligned}\right\}$$

The above describes a particular travelling plane of the wave $\mathcal{W}$ evaluated at the position $\mathbf{p}$ for any initial position $-\infty < z_i < \infty$ at $t = 0$. It is the classic description for a travelling plane wave with phase velocity defined by vector $\mathbf{u}$. But why so complicated? With $p = z_i + ct$ we can simplify the above to

$$\mathcal{W}(\mathbf{p}_i, t) \xrightarrow[\text{by}]{\text{par}} \left\{ \begin{aligned} &\mathbf{u} = \hat{z}c \\ &\mathbf{B} = B\big[\hat{x}\cos(\omega z_i/c) + \hat{y}\sin(\omega z_i/c)\big] \\ &\mathbf{E} = cB\big[-\hat{x}\sin(\omega z_i/c) + \hat{y}\cos(\omega z_i/c)\big] \end{aligned} \right\} \tag{14}$$

and we know that the above parameters provide a solution to $\mathcal{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ and I have shown that $\mathcal{M}$ is a reformulation of the Maxwell equation in vacuum. Therefore (14) describes a particular plane of an EM-travelling plane wave.

It is well known, and corroborated by experience, that radio waves are described by the above. A radio wave is a train of infinitely many planes that make up the continuous transmitted signal. Here we must note that each travelling plane is static; nothing changes within it whilst propagating.

2.2 Circular EM*-Soliton*

If the parameter equations $\mathcal{P}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ provide a solution to $\mathcal{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ then $\mathcal{P}$ describes an EM-wave. That the vector $\mathbf{u}$ is one dimensional is a choice and not a necessity; we can define the velocity vector $\mathbf{u} = c(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$, that is a circular path given by $\mathbf{p} = \int \mathbf{u}\,\mathrm{d}t = (c/\omega)\big(\hat{x}\sin\omega(t_i + t) + \hat{y}\cos\omega(t_i + t)\big)$. The circular EM-solton Γ is defined by

$$\Gamma(\mathbf{p}, t) \xrightarrow[\text{by}]{\text{par}} \left\{ \begin{aligned} &\mathbf{u} = c\big(\hat{x}\cos\omega(t_i + t) - \hat{y}\sin\omega(t_i + t)\big) \\ &\mathbf{B} = \hat{z}B\cos n\omega t_i \\ &\mathbf{E} = -cB\cos n\omega t_i\big(\hat{x}\sin\omega(t_i + t) + \hat{y}\cos\omega(t_i + t)\big) \end{aligned} \right\} \tag{15}$$

where n is an integer. To understand the above, let's construct a table that lists the above quantities for ωt_i, $t = 0$ and $n = 1$.

i	1	2	3	4	5	6
ωt_i	0	$\pi/4$	$\pi/2$	π	$5\pi/4$	$3\pi/2$
$\mathbf{u}$	$\hat{x}c$	$\sqrt{½}(\hat{x} - \hat{y})c$	$-\hat{y}c$	$-\hat{x}c$	$\sqrt{½}(-\hat{x} - \hat{y})c$	$\hat{y}c$
$\mathbf{p}$	$\hat{y}r$	$\sqrt{½}(\hat{x} + \hat{y})r$	$\hat{x}r$	$-\hat{y}r$	$\sqrt{½}(\hat{x} - \hat{y})c$	$-\hat{x}r$
$\mathbf{B}$	$\hat{z}B_0$	$\sqrt{½}\hat{z}B_0$	0	$-\hat{z}B_0$	$-\sqrt{½}\hat{z}B_0$	0
$\mathbf{E}$	$-\hat{y}cB_0$	$-\sqrt{½}(\hat{x} + \hat{y})cB_0$	0	$\hat{y}cB_0$	$-\sqrt{½}(\hat{x} - \hat{y})cB_0$	0

where $r = c/\omega$. The above table suggests that the flux of the magnetic field of planes 1&4 are one and the same and form a loop, the same applies for any two planes opposed diametrically. We also note that the electric field vector $\mathbf{E}$ always points to the center of the circular path. That suggests that a stationary point on the circular path experiences electric potential varying between zero and some potential V volts, but always of the same polarity. We thus can interpret (15) as a stationary EM-soliton, made up of infinitely many planes rotating around a central point. The magnetic flux loops connecting opposing planes, *e.g.* 1&4, would collapse to $r \to 0$, a lower energetic state, were it not for the equal potentials at the diametrically opposite sides of the magnetic loop that generate an opposing force preventing its collapse.

2.3 Proposition: Ball lightning as a three dimensional EM-soliton

Accounts of ball lightning are reported on a regular basis yet all scientific explanations have evaded general acceptance. Here I propose another explanation that ball lightning is an EM-soliton; a wave propagating on a wound up near spherical path predicted my the equation set $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$. For a travelling object w we define a unit velocity vector as

$$\hat{\mathrm{u}}_w(t) = \hat{\mathrm{x}}\sin\omega_1 t\sin\omega_2 t + \hat{\mathrm{y}}\sin n\omega_1 t\cos\omega_2 t + \hat{\mathrm{z}}\cos\omega_1 t$$

The path $\mathbf{s}_w$ that the object w follows is found by integration

$$\begin{aligned}\mathbf{s}_w(t) = \int c\hat{\mathrm{u}}_w\,\mathrm{d}t = \hat{\mathrm{x}}c\left(\frac{\sin(\omega_2+\omega_1)t}{2(\omega_2+\omega_1)} - \frac{\sin(\omega_1-\omega_2)t}{2(\omega_1-\omega_2)}\right)\\ +\hat{\mathrm{y}}c\left(\frac{\cos(\omega_2+\omega_1)t}{2(\omega_2+\omega_1)} + \frac{\cos(\omega_1-\omega_2)t}{2(\omega_1-\omega_2)}\right) - \hat{\mathrm{z}}c\frac{\sin\omega_1 t}{\omega_1}\end{aligned} \tag{16}$$

The path is closed, or repeats, in periods of $t = 2\pi$ and as $\|c\hat{\mathrm{u}}_w(t)\| = c$ the pathlength of $\mathbf{s}_w$ is $2\pi c$. Figure-2 sketches examples of paths defined by the above with different combinations of ω_1 and ω_2.

Let's define an EM-soliton Θ as a EM-wave that exists only on the closed path $\mathbf{p} = \int c\hat{\mathrm{u}}_w\,\mathrm{d}t$, and at $t = 0$ it is at the position $\mathbf{p}_0 = \mathbf{s}_w(t_0)$, thus $\mathbf{u} = c\,\hat{\mathrm{u}}_w(t_0+t)$, hence:

$$\Theta(\mathbf{p},t) \xrightarrow[\text{by}]{\text{dsc}} \left\{\begin{aligned} &\mathbf{u} = c\big(\hat{\mathrm{x}}\sin\omega_1(t_0+t)\sin\omega_2(t_0+t)\\ &\qquad +\hat{\mathrm{y}}\sin n\omega_1(t_0+t)\cos\omega_2(t_0+t) + \hat{\mathrm{z}}\cos\omega_1(t_0+t)\big)\\ &\mathbf{B} = B\big(\hat{\mathrm{x}}\cos\omega_2(t_0+t) - \hat{\mathrm{y}}\sin\omega_2(t_0+t)\big)\\ &\mathbf{E} = cB\big(\hat{\mathrm{x}}\cos\omega_1(t_0+t)\sin\omega_2(t_0+t)\\ &\qquad +\hat{\mathrm{y}}\cos\omega_1(t_0+t)\cos\omega_2(t_0+t) - \hat{\mathrm{z}}\sin\omega_1(t_0+t)\big)\end{aligned}\right\} \tag{17}$$

The above satisfies the equation set $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$. Therefore, the above describes an EM-wave; propagating on the closed path $\mathbf{p}$ at velocity c. The above equation set (17) suggests that the wave is "trapped" by its magnetic field which forms a closed ring that precesses with the wave motion, that is when connected to the magnetic field of a point retarded by $\omega_2(t_0 - ½)$ but at a different z-elevation; and with the electric field always radiating outwards. Here I need to point out that Arnhoff

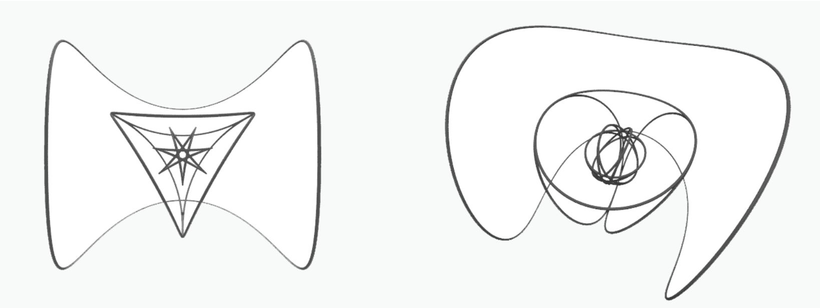

Figure 2: Two views of the path $\hat{s}_w(t)$ defined by (16) for the frequency ratios $\omega_1 : \omega_2 = 1:2$, $1:3$, and $1:7$. The path length of each curve is 2π.

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[8], Chubykalo [9] and Cameron [10] presented solutions for three dimensional EM-wave structures, all of which are based on the superposition principle which allows the construction of intricate wave structures, and all contrasting with the simplicity of (17). Boerner [11] considers Cameron's proposal as the only viable explanation for ball lightning.

The term *soliton* describes self reinforcing solitary waves. Drazin [12] defined a soliton as any solution of a nonlinear equation (or a system) which:

i. *represents a wave of permanent form;*
ii. *is localised, so that it decays or approaches a constant at infinity;*
iii. *can interact strongly with other solitons and retain its identity.*

The above analysis confirms the first two points; the third is yet to be demonstrated.

3 Concluding remark

With this article I present a novel wave equation system, in which each possible solution describes a bimodal-transverse wave in a more *aufschlussreicher*[1] way than was possible with the classic partial differential approach. Remarkably, this equation set proves to be fundamental to electromagnetic theory as it demands the formulations for ϵ_0 and μ_0 in a form previously only derivable from seemingly unrelated atomic theories and physical observations.

Whether or not the new wave equation set $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ gets general acceptance is not for me to determine; if it does then undoubtedly many derivations of it will

1 aufschlussreich: German adj., translations: enlightening, illuminating, informative, insightful, instructive, revealing, and telling.

be developed. Whether or not it provokes a rethinking of the electromagnetic phenomenon, or whether new discoveries are made resulting from all of the above, only time will tell. Nevertheless—for me—this paper marks the beginning of new work in this subject. There is much that remains to be done; for example, extending the methods developed here to model particles as waves. I have developed an interesting Ansatz, but to bring it to conclusion requires some collaborative effort and intellectual sparring partners to review, critique and contribute towards an extended and collaborative work.

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