A new n-dimensional associative and commutative, but non-distributive, algebra

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Abstract: This paper introduces a new n-dimensional associative and commutative, but non-distributive, algebra. We define the spatial operator ${}_{\mathrm{s}}\mathrm{j}$ that manipulates numbers in a multidimensional number-space (hyper-complex) according to the spatial angle ${}_{\mathrm{s}}\theta$, a tuple of angles ${}_{\mathrm{s}}(\theta_1,\theta_2,\ldots)$. The spatial number, which is expressed symbolically as $\mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}\theta}$, belongs to both the additive and multiplicative Abelian groups. They are non-distributive in multiplication with respect to addition, thus forming a non-distributive ring. Spatial numbers could have applications in vector algebra allowing the algebraic product of two vector quantities. Furthermore, they could be of interest in physics, and towards that purpose, I present a novel multi-dimensional solution of the wave equation that describes a spherical wave object whose centre propagates at a velocity c in a vector space.

This paper introduces the spatial numbers of infinite dimensionality

$$\begin{aligned}\mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}\theta} = \mathrm{j}_0 \cos\theta_0 \cos\theta_1 \cos\theta_2 \ldots \\ + \mathrm{j}_1 \sin\theta_0 \cos\theta_1 \cos\theta_2 \ldots \\ + \mathrm{j}_2 \sin\theta_1 \cos\theta_2 \ldots + \cdots + \mathrm{j}_k \sin\theta_k\end{aligned}$$

or

$$r\,\mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}(\theta_1,\theta_2,\theta_3,\ldots,\theta_k)} = r\left(\mathrm{j}_0 \prod_{z=1}^{k} \cos\theta_z + \sum_{x=1}^{k}\left(\mathrm{j}_x \sin\theta_x \prod_{y=-1}^{k-x-1} \cos\big(\theta_{(k-y)}\big)\right)\right) \tag{1}$$

where $\mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}\theta}$ is a symbolic representation for consecutive orthogonal rotations the result being a multidimensional number, ${}_{\mathrm{s}}\mathrm{j}$ is the spatial operator, ${}_{\mathrm{s}}\theta$ is the spatial angle defined by a tuple of angles ${}_{\mathrm{s}}\theta = {}_{\mathrm{s}}(\theta_1,\theta_2,\theta_3,\ldots\theta_k)$, and j_n is the spatial unit orthogonal to all other j_m, $0 \le m \le k$ and $m \ne n$. The spatial number $\mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}\theta}$ could be viewed as a hypercomplex number. For a spatial number of rank k, the angles $\theta_1,\theta_2,\ldots\theta_k$ are defined, and $\theta_{(l>k)} = 0$. The expansions to rank three are:

Symbol	Description
$\mathbb{R}$	The field of real numbers
e	A symbol and not the natural number $e = 2.71828\ldots$
i	The imaginary unit number, or imaginary operator.
$\mathbb{C}$	The field of complex numbers $z = re\theta$.
${}_{\mathrm{s}}\mathrm{j}$	The preceding subscript 's' indicates a spatial 'thing', which is either the spatial constructor ${}_{\mathrm{s}}\mathrm{j}$ or a spatial angle ${}_{\mathrm{s}}\theta$ (see below). The spatial constructor is characterised by ${}_{\mathrm{s}}\mathrm{j}^2 = -1$, and it implies a number with dimensionality.
j_n	The spatial constructor ${}_{\mathrm{s}}\mathrm{j}$ implies a multi-dimensional number space $\mathcal{S}$ whose axes we label, and which are defined by the spatial units $\mathrm{j}_0, \mathrm{j}_1, \mathrm{j}_2, \mathrm{j}_3, \ldots$, with $\mathrm{j}_0 = 1$.
$\mathcal{S}$, ${}^k\mathcal{S}$, $[\![\,]\!]$	An Euclidian spatial-number space $\mathcal{S}$, each axis is labelled by the associated spatial unit. $\mathcal{S}$ can also be expressed by enclosing the spatial units in double struck square brackets ${}^k\mathcal{S} = [\![\mathrm{j}_0, \mathrm{j}_1, \mathrm{j}_2, \ldots \mathrm{j}_k]\!]$.
${}_{\mathrm{s}}\theta$, ${}_{\mathrm{s}}()$	The spatial angle ${}_{\mathrm{s}}\theta$ defined as a tuple consisting of many rotation angles ${}_{\mathrm{s}}(\theta_1, \theta_2, \theta_3, \ldots)$, each towards an axis j_n which is orthogonal to all previous axes j_m and $0 \le m < n$. (${}_{\mathrm{s}}\theta$ is pronounced as spatial theta.)
${}^k_{\mathrm{s}}\theta$	The preceding superscript 'k', usually a number, limits the rank of a spatial variable to 'k'. *E.g.* ${}^3_{\mathrm{s}}\theta = {}_{\mathrm{s}}(\theta_1, \theta_2, \theta_3)$. Similarly ${}^k_{\mathrm{s}}\mathrm{j} \Rightarrow (\mathrm{j}_0, \mathrm{j}_1, \mathrm{j}_2, \ldots \mathrm{j}_k)$. (An angle θ_0 does not exist.)
$\mathbb{V}$	A non-distributive ring of spatial numbers $v = r\,\mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}\alpha}$.
$\mathring{a}$	The circle accent (reminding one of spheres) is an optional notation when one wishes to emphasise that the number is a spatial number, *e.g.* $\mathring{a} = \mathrm{e}^{{}_{\mathrm{s}}\mathrm{j}\,{}_{\mathrm{s}}\theta}$.
${}^k\mathring{a}$	The rank of a spatial number is predetermined, *e.g.* ${}^2\mathring{a} = \mathrm{j}_0 a_0 + \mathrm{j}_1 a_1 + \mathrm{j}_2 a_2$.
$\oplus$, $\ominus$	The binary operators for adding, or subtracting, orthogonal rotations
$\otimes$	The binary operator for multiplying orthogonally rotated quantities.

NOTE The semantics *rotate and rotation* in conjunction with numbers is best explained as follows: Using complex numbers as an example we have no problem in understanding the complex plane as a two dimensional number plane. The number $z = re^{i\theta}$ could be described as rotating the number r off the real axis towards the imaginary axis by an angle theta, that is we are rotating a line defined by zero and r both on the real axis. After this rotation we have $z = r\cos\theta + ir\sin\theta$.

$$\begin{aligned}
k=0:&\quad \mathrm{e}^{{}_\mathrm{s}\mathrm{j}0} = 1\\
k=1:&\quad \mathrm{e}^{{}_\mathrm{s}\mathrm{j}{}_\mathrm{s}(\theta_1)} = \mathrm{j}_0\cos\theta_1 + \mathrm{j}_1\sin\theta_1\\
k=2:&\quad \mathrm{e}^{{}_\mathrm{s}\mathrm{j}{}_\mathrm{s}(\theta_1,\theta_2)} = \mathrm{j}_0\cos\theta_1\cos\theta_2 + \mathrm{j}_1\sin\theta_1\cos\theta_2 + \mathrm{j}_2\sin\theta_2\\
k=3:&\quad \mathrm{e}^{{}_\mathrm{s}\mathrm{j}{}_\mathrm{s}(\theta_1,\theta_2,\theta_3)} = \mathrm{j}_0\cos\theta_1\cos\theta_2\cos\theta_3\\
&\qquad + \mathrm{j}_1\sin\theta_1\cos\theta_2\cos\theta_3\\
&\qquad\quad + \mathrm{j}_2\sin\theta_2\cos\theta_3 + \mathrm{j}_3\sin\theta_3
\end{aligned}$$

These numbers remind one of spherical coordinates. *E.g.* for rank $k=2$, θ_1 is the azimith and θ_2 the elevation as is the case in the Earth's mapping that uses longitude and latitude, but unlike in spherical coordinate systems, the angles are not limited to a specific range.

Theorem 1: *The spatial numbers $\mathring{v} = r\mathrm{e}^{{}_\mathrm{s}\mathrm{j}{}_\mathrm{s}\theta}$ belong to both the additive and multiplicative Abelian groups, and form a non-distributive ring, which we symbolise with $\mathbb{V}$. Hence, an associative and commutative, but non distributive, n-dimensional algebra emerges.*

This algebra, although severely limited could have speciality applications. Mapping the spatial number into a vector space $[\mathrm{j}_0, \mathrm{j}_1, \mathrm{j}_2] = [\mathrm{X},\mathrm{Y},\mathrm{Z}]$ allows the algebraic products of vector quantities. In physics they could be used to find novel multi-dimensional solutions to the wave equation as elucidated in a later section, which could help in describing elementary particles.

A spatial number is expressed in the Euler form as

$$\mathring{v} = r\mathrm{e}^{{}_\mathrm{s}\mathrm{j}{}_\mathrm{s}\theta}$$

where $r \in \mathbb{R}$, ${}_\mathrm{s}\mathrm{j}$ is the spatial constructor, and ${}_\mathrm{s}\theta = {}_\mathrm{s}(\theta_1,\theta_2,\theta_3,\ldots)$. The product ${}_\mathrm{s}\mathrm{j}{}_\mathrm{s}\theta$ expands to

$${}_\mathrm{s}\mathrm{j}{}_\mathrm{s}\theta = \mathrm{j}_1\theta_1 \oplus \mathrm{j}_2\theta_2 \oplus \mathrm{j}_3\theta_3 \oplus\cdots\oplus \mathrm{j}_k\theta_k \tag{2}$$

where $\mathrm{j}_1\theta_1$ means rotate from the root axis j_0 towards the j_1-axis by θ_1, and $\oplus\mathrm{j}_2\theta_2$ means rotate the result of the previous rotation from the j_0–j_1 plane towards j_2-axis by θ_2, etc. Therefore

$$\mathring{v} = \mathrm{e}^{{}_\mathrm{s}\mathrm{j}{}_\mathrm{s}\theta} = e^{\mathrm{j}_1\theta_1\oplus\mathrm{j}_2\theta_2\oplus\mathrm{j}_3\theta_3\oplus\cdots\oplus\mathrm{j}_k\theta_k} \tag{3a}$$

$$= e^{\mathrm{j}_1\theta_1} \otimes e^{\mathrm{j}_2\theta_2} \otimes e^{\mathrm{j}_3\theta_3} \otimes\cdots\otimes e^{\mathrm{j}_k\theta_k} \tag{3b}$$

where each term $e^{\mathrm{j}_n\theta_n}$ is evaluated like a complex number with a ranked imaginary number j_n. The operators $\otimes$, and $\oplus$ are used only in the above context to explain the relation between the Euler and rectilinear forms of spatial numbers.

The orthogonal product $e^{\mathrm{j}_1\theta_1} \otimes e^{\mathrm{j}_2\theta_2}$ is defined by

$$
\begin{aligned}
e^{\mathrm{j}_1\theta_1} \otimes e^{\mathrm{j}_2\theta_2} &= (\mathrm{j}_0 \cos\theta_1 + \mathrm{j}_1 \sin\theta_1) \otimes (\cos\theta_2 + \mathrm{j}_1 \sin\theta_2) && \text{(4a)}\\
&= (\mathrm{j}_0 \cos\theta_1 + \mathrm{j}_1 \sin\theta_1)(\cos\theta_2) + |\mathrm{j}_0 \cos\theta_1 + \mathrm{j}_1 \sin\theta_1|(\mathrm{j}_1 \sin\theta_2) && \text{(4b)}\\
&= (\mathrm{j}_0 \cos\theta_1 \cos\theta_2 + \mathrm{j}_1 \sin\theta_1 \cos\theta_2) + \mathrm{j}_1 \sin\theta_2 && \text{(4c)}
\end{aligned}
$$

and we note that in (4b) the $\mathrm{j}_1 \sin\theta_2$ term multiplies with the absolute value $\left|\mathrm{j}_0 \cos\theta_1 + \mathrm{j}_1 \sin\theta_1\right|$ or generalised

$$
\begin{aligned}
\mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} \otimes e^{\mathrm{j}_{(k+1)}\theta_{(k+1)}} &= \mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} \cos\theta_{(k+1)} + \left|\mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta}\right|(\mathrm{j}_{(k+1)} \sin\theta_{(k+1)}) && \text{(5a)}\\
&= \mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} \cos\theta_{(k+1)} + \mathrm{j}_{(k+1)} \sin\theta_{(k+1)} && \text{(5b)}
\end{aligned}
$$

Let $\mathring{a} = r_a \mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}\alpha}$ and $\mathring{b} = r_b \mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}\beta}$. Analogous to complex numbers, the product of two spatial numbers

$$
\mathring{a}\mathring{b} = r_a r_b \mathrm{e}^{\mathrm{sj}({}_{\mathrm{s}}\alpha + {}_{\mathrm{s}}\beta)} \tag{6}
$$

where $({}_{\mathrm{s}}\alpha + {}_{\mathrm{s}}\beta) = {}_{\mathrm{s}}\left(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, \ldots\right)$ *i.e.* the simple summing of the angles. The sum of two spatial numbers is over the rectilinear coefficients

$$
\mathring{a} + \mathring{b} = \mathrm{j}_0(a_0 + b_0) + \mathrm{j}_1(a_1 + b_1) + \mathrm{j}_2(a_2 + b_2) + \ldots \tag{7}
$$

Unfortunately, the distributivity over multiplication is lost for rank $k \geq 2$

$$
({}^{k}\mathring{a} + {}^{k}\mathring{b})\,{}^{k}\mathring{c} \neq {}^{k}\mathring{a}\,{}^{k}\mathring{c} + {}^{k}\mathring{b}\,{}^{k}\mathring{c} \quad \text{if} \quad k \geq 2 \tag{8}
$$

Proof: For above theorem we require (6),(7) and (8) to be true.

Associativity and commutativity of multiplication: To multiply two complex numbers we need one rule $\mathrm{i}^2 = -1$, or we can use the Euler rule of summing the angles $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta} = \mathrm{e}^{\mathrm{i}\alpha+\beta}$. Similarly, to obtain the product of two spatial numbers, we can simply sum the spatial angles $\mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}\alpha}\mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}\beta} = \mathrm{e}^{\mathrm{sj}({}_{\mathrm{s}}\alpha + {}_{\mathrm{s}}\beta)}$, or we can use a procedure using a mix of the Euler and Euclidean forms and the following rules:

$$
\begin{aligned}
\mathrm{j}_0^2 &= \mathrm{j}_0 = 1 && \text{(9a)}\\
\mathrm{j}_1^2 &= -\,\mathrm{j}_0 = -1 && \text{(9b)}\\
\mathrm{j}_{(k+1)}^2 &= -\,\mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} = -\mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}(\theta_1, \theta_2, \theta_3, \ldots \theta_k)} && \text{(9c)}\\
\mathrm{j}_{(k+1)}^3 &= -\,\mathrm{j}_{(k+1)} \quad \text{therefore} \quad \mathrm{j}_{(k+1)}\mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} = \mathrm{j}_{(k+1)} && \text{(9d)}\\
{}_{\mathrm{s}}\mathrm{j}^2 &= -\,1 && \text{(9e)}
\end{aligned}
$$

We explain rule (9c) by using (5)

$$
\begin{aligned}
\mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} \otimes e^{\mathrm{j}_{(k+1)}\pi/2} &= \mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta} \sin\pi/2 + \left|\mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta}\right| \mathrm{j}_{(k+1)} cos\pi/2 \\
&= \mathrm{j}_{(k+1)}
\end{aligned}
$$

As per definition $\mathrm{j}_{(k+1)}$ can only exist if ${}^{k}\mathring{v} = \mathrm{e}^{\mathrm{sj}\,{}^{k}_{\mathrm{s}}\theta}$ exists. We are rotating a spatial number ${}^{k}\mathring{v}$ off its k^{th}-dimensional plane towards the $(k+1)^{\text{th}}$-axis. A 90^{o} rotation results in a number with just a $\mathrm{j}_{(k+1)}$ component and all other j_n components, $0 \le n \le k$, being zero. If the rotation is 180^{o} then ${}^{k}\mathring{v}$ is negated.

That multiplication is a simple addition of the rotation angles also needs to be demonstrated in the cartesian form: Let $\hat{a}$ and $\hat{b}$ be unit spatial numbers with rotation angle tuples ${}_{\mathrm{s}}(\alpha_1,\alpha_2,\dots\alpha_n)$ and ${}_{\mathrm{s}}\big(\beta_1,\beta_2,\dots,\beta_n\big)$ respectively. We express $\hat{a}$ and $\hat{b}$ as a mix of the Euler form and rectilinear form and demonstrate:

$$k=0 \;:\; \hat{a}\hat{b} = \mathrm{j}_0^2 = 1 \tag{10}$$

$$k=1 \;:\; \hat{a}\hat{b} = \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1)}\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\beta_1)} = \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\beta_1)} \tag{11}$$

$$k=2 \;:\; \hat{a}\hat{b} = \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\alpha_2)}\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\beta_1+\beta_2)} \tag{12}$$

$$
\begin{aligned}
&= (\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1)}\cos\alpha_2 + \mathrm{j}_2\sin\alpha_2) && \text{(a)}\\
&\qquad \times (\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\beta_1)}\cos\beta_2 + \mathrm{j}_2\sin\beta_2) && \text{(b)}\\
&= \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha 1+\beta_1)}\cos\alpha_2\cos\beta_2 && \text{(c)}\\
&\quad + \mathrm{j}_2\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1)}\cos\alpha_2\sin\beta_2 && \text{(d)}\\
&\quad + \mathrm{j}_2\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\beta_1)}\sin\alpha_2\cos\beta_2 && \text{(e)}\\
&\quad + \mathrm{j}_2^2\sin\alpha_2\sin\beta_2 && \text{(f)}\\
&= \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\beta_1)}\cos\big(\alpha_2+\beta_2\big) + \mathrm{j}_2\sin\big(\alpha_2+\beta_2\big) && \text{(g)}\\
&= \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\beta_1,\alpha_2+\beta_2)} && \text{(h)}
\end{aligned}
$$

therefore by induction:

$$k=n \;:\; \hat{a}\hat{b} = \mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\beta_1,\alpha_2+\beta_2,\dots\alpha_n+\beta_n)} \tag{13}$$

The Equations (10 and 11) need no comment, (11) are the complex numbers. Equation (12) is the product of two spatial numbers of order two (or three dimensions) and (a) and (b) expresses the product as a mix of spatial Euler and rectilinear form. This expands into four parts (c) though (f). We combine parts (c) and (f) and we note by Rule (9b) for the product $\hat{a}\hat{b}$ that $\mathrm{j}_2^2 = -\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\beta_1)}$, thus parts (c) and (f) have reduced to $\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1+\beta_1)}\cos\big(\alpha_2+\beta_2\big)$ the first term in (g). Examining parts (d) and (e), the terms $\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1)}$ and $\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\beta_1)}$ only indicate a position from which a rotation towards j_2 took place and have no influence on the value $a_2\mathrm{j}_2$ and $b_2\mathrm{j}_2$ respectively, thus both $\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\alpha_1)}$ and $\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\beta_1)}$ can be replaced by their respective absolute value, which in both cases is one. Thus combining parts (d) and (e) yields the second term $\mathrm{j}_2 \sin\left(\alpha_2+\beta_2\right)$ in (g), and trigonometrical reduction yields (h) *i.e.* the product $\hat{a}\hat{b}$ in the spatial Euler notation.

We have now shown that the product of two spatial numbers is indeed the sum of the spatial angles; thus the associativity and commutativity of multiplication is given.

Associativity and commutativity of addition is given; trivial by addition of the rectilinear coefficients and elucidated in Appendix A.

Additive and multiplicative identity: Trivial; the additive and multiplicative identities are 0 and 1 respectively.

Additive inverses: Trivial; for every ${}^k\mathring{v} = \mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}(\theta_1,\theta_2,\ldots,\theta_k)}$ there exists a ${}^k\mathring{w} = \mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}(\theta_1,\theta_2,\ldots,\theta_k+\pi)}$, which yields the sum ${}^k\mathring{v} + {}^k\mathring{w} = 0$, therefore $-{}^k\mathring{v} = {}^k\mathring{w}$.

Multiplicative inverses: Trivial; for every ${}^k\mathring{v} = \mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}(\theta_1,\theta_2,\ldots,\theta_k)}$ there exists a ${}^k\mathring{w} = \mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}(-\theta_1,-\theta_2,\ldots,-\theta_k)}$, which yields the product ${}^k\mathring{v}\,{}^k\mathring{w} = 1$, therefore ${}^k\mathring{v}^{-1} = {}^k\mathring{w}$.

All requirements for both the additive and multiplicative Abelian groups are satisfied.

Non distributivity of multiplication over addition for rank $k \geq 2$: Trivial; demonstrated by numeric evaluation, and also elucidated in Appendix B.

$\square$

Integration and differentiation

Of particular interest is to integrate and differentiate the spatial number. The derivatives and integrals for spatial numbers are basically the same as those for complex numbers.

$$\frac{\mathrm{d}}{\mathrm{d}_\mathrm{s}\theta}\left({}_\mathrm{s}\mathrm{j}\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta}\right) = {}_\mathrm{s}\mathrm{j}\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta} \qquad \text{and} \qquad \int {}_\mathrm{s}\mathrm{j}\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta}\,\mathrm{d}_\mathrm{s}\theta = -{}_\mathrm{s}\mathrm{j}\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta} \tag{14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}_\mathrm{s}\mathrm{j}\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta t}\right) = {}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta t} \qquad \text{and} \qquad \int \mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta t}\,\mathrm{d}t = -{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta^{-1}\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta t} \tag{14b}$$

But how to evaluate ${}_\mathrm{s}\mathrm{j}$ or ${}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta$? They are not numbers ${}_\mathrm{s}\mathrm{j}$ is a constructor that uses the spatial angle ${}_\mathrm{s}\theta$. This becomes clear when evaluating the first and second derivative of $\varphi = 2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}\omega t} = 2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}_\mathrm{s}(\omega_1,\omega_2)t}$ with respect to t, and using trigonometric expansion formulas.

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(2\mathrm{j}_0\cos\omega_1 t\cos\omega_2 t + 2\mathrm{j}_1\sin\omega_1 t\cos\omega_2 t + 2\mathrm{j}_2\sin\omega_2 t\big) \tag{15a}$$

$$\begin{aligned} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(\mathrm{j}_0\big(\cos(\omega_1-\omega_2)t + \cos(\omega_1+\omega_2)t\big) \\ &\qquad + \mathrm{j}_1\big(\sin(\omega_1-\omega_2)t + \sin(\omega_1+\omega_2)t\big) \\ &\qquad + 2\mathrm{j}_2\sin\omega_2 t\Big) \end{aligned} \tag{15b}$$

$$\begin{aligned} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(-\mathrm{j}_0\big((\omega_1-\omega_2)\sin(\omega_1-\omega_2)t + (\omega_1+\omega_2)\sin(\omega_1+\omega_2)t\big) \\ &\qquad + \mathrm{j}_1\big((\omega_1-\omega_2)\cos(\omega_1-\omega_2)t + (\omega_1+\omega_2)\cos(\omega_1+\omega_2)t\big) \\ &\qquad + 2\mathrm{j}_2\omega_2\cos\omega_2 t\Big) \end{aligned} \tag{15c}$$

$$\begin{aligned} &= -\mathrm{j}_0\big((\omega_1-\omega_2)^2\cos(\omega_1-\omega_2)t + (\omega_1+\omega_2)^2\cos(\omega_1+\omega_2)t\big) \\ &\qquad - \mathrm{j}_1\big((\omega_1-\omega_2)^2\sin(\omega_1-\omega_2)t + (\omega_1+\omega_2)^2\sin(\omega_1+\omega_2)t\big) \\ &\qquad - 2\mathrm{j}_2\omega_2^2\sin\omega_2 t \end{aligned} \tag{15d}$$

$$= {}_\mathrm{s}\mathrm{j}^2{}_\mathrm{s}(\omega_1,\omega_2)^2\, 2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}\,{}_\mathrm{s}(\omega_1,\omega_2)\,t} = -{}_\mathrm{s}\omega^2 2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}\,{}_\mathrm{s}\omega t} \tag{15e}$$

$$= {}_\mathrm{s}\mathrm{j}^2{}_\mathrm{s}(\omega_1,\omega_2)^2\,\varphi = -{}_\mathrm{s}\omega^2\varphi \tag{15f}$$

From Equations (15a) to (15f) we understand that ϕ describes an undamped harmonic oscillator in three dimensions, consisting of five sub harmonic oscillators in superposition. The last two lines, (15e) and (15f) are short hand conventions to describe (15d). When working with spatial numbers we can treat the constructs ${}_\mathrm{s}\mathrm{j}$, ${}_\mathrm{s}\mathrm{j}_\mathrm{s}\omega$ and $-{}_\mathrm{s}\omega^2$ symbolically as numbers as long as we understand the expansion behind these constructs. The expressions ${}_\mathrm{s}\mathrm{j}\,x$ or ${}_\mathrm{s}\mathrm{j}_\mathrm{s}\omega\,x$, with $x \in \{\mathbb{R},\mathbb{C}\}$, are meaningless.

To answer the earlier question "*how to evaluate* ${}_\mathrm{s}\mathrm{j}$ *or* ${}_\mathrm{s}\mathrm{j}_\mathrm{s}\theta$*?*" From (14b) and (15c), and with $\varphi = 2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}\,{}_\mathrm{s}\omega t} = 2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}\,{}_\mathrm{s}(\omega_1,\omega_2)\,t}$ we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = {}_\mathrm{s}\mathrm{j}_\mathrm{s}(\omega_1,\omega_2)\,2\mathrm{e}^{{}_\mathrm{s}\mathrm{j}\,{}_\mathrm{s}(\omega_1,\omega_2)\,t} \tag{16a}$$

$$\begin{aligned} &= -\mathrm{j}_0\big((\omega_1-\omega_2)\sin(\omega_1-\omega_2)t + (\omega_1+\omega_2)\sin(\omega_1+\omega_2)t\big) \\ &\qquad + \mathrm{j}_1\big((\omega_1-\omega_2)\cos(\omega_1-\omega_2)t + (\omega_1+\omega_2)\cos(\omega_1+\omega_2)t\big) \\ &\qquad + 2\mathrm{j}_2\omega_2\cos\omega_2 t \end{aligned} \tag{16b}$$

Therefor, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}_{\mathrm{s}}\omega}\left(2\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\omega_1,\omega_2)}\right) = {}_{\mathrm{s}}\mathrm{j}2\mathrm{e}^{\mathrm{sj}\,\mathrm{s}(\omega_1,\omega_2)} \tag{17a}$$

$$\begin{aligned} &= -\mathrm{j}_0\big(\sin(\omega_1-\omega_2)+\sin(\omega_1+\omega_2)\big) \\ &\quad + \mathrm{j}_1\big(\cos(\omega_1-\omega_2)+\cos(\omega_1+\omega_2)\big) \\ &\quad + 2\mathrm{j}_2\cos\omega_2 \end{aligned} \tag{17b}$$

A multi-dimensional solution to the wave equation.

Spatial numbers allow a novel solution of the wave equation, a second order partial differential equation of form

$$c^2\frac{\partial^2\mathscr{W}}{\partial p^2} - \frac{\partial^2\mathscr{W}}{\partial t^2} = 0 \tag{18}$$

where in the spatial number form $\mathscr{W}$ is a wave structure centred at a position p in a vector space [XYZ], c the speed at which the wave structure propagates, and t is time. To solve (18), we need separate time and position components for $\mathscr{W}$, which is simply achieved by

$$\mathscr{W} = \begin{cases} \mathscr{P}(p) = \mathscr{X}^2(p) \\ \mathscr{T}(t) = \mathscr{Y}^2(t) \\ \mathscr{X}(p)\mathscr{Y}(t) \end{cases} \tag{19}$$

from which quickly follows

$$\frac{c^2}{\mathscr{X}(p)}\frac{\mathrm{d}^2\mathscr{X}(p)}{\mathrm{d}p^2} = \frac{1}{\mathscr{Y}(t)}\frac{\mathrm{d}^2\mathscr{Y}(t)}{\mathrm{d}t^2} \quad = -\frac{{}_{\mathrm{s}}\omega^2}{4} \tag{20}$$

where the right hand term is introduced in anticipation of the desired result. We also note that $\mathscr{W},\mathscr{P},\mathscr{T},\mathscr{X},\mathscr{Y} \in \mathbb{V}$ also satisfies (19) and (20). The derivatives are total as $\mathscr{X}(p)$ and $\mathscr{Y}(t)$ are independent of one another, but also equal to each other if and only if

$$p = p_o + \hat{\kappa}ct \tag{21}$$

where p_o is some initial position and $\hat{\kappa}$ is a unit direction vector. The solutions of the two second order differential equations using spatial numbers was demonstrated in (15), hence

$$\mathscr{X}(p) = \sqrt{A}\mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}\omega p/2c+{}_{\mathrm{s}}\theta_o/2} \tag{22a}$$

$$\mathscr{Y}(t) = \sqrt{A}\mathrm{e}^{\mathrm{sj}\,{}_{\mathrm{s}}\omega t/2++{}_{\mathrm{s}}\theta_o/2} \tag{22b}$$

where ${}_{\mathrm{s}}\theta_o$ is an arbitrary initial condition, and A is an arbitrary constant independent of time or position and characterises the wave structure $\mathscr{W}$. All that remains is to square $\mathscr{Y}(t)$ and taking care of initial conditions to obtain

$$\mathscr{W} = A\mathrm{e}^{\mathrm{sjs}\omega t + {}_{\mathrm{s}}\theta_o} \tag{23}$$

Therefore, in the vector space [XYZ], the wave structure $\mathscr{W}$ has spherical properties. It is a mathematical object that is described by the wave equation (18) centred around a point p that propagates with a velocity c.

Appendix A; Associativity and commutativity of addition

Let ${}^k\mathring{a} = r_a \mathrm{e}^{\mathrm{sjs}(\alpha_1,\alpha_2,\ldots,\alpha_k)}$, ${}^k\mathring{b} = r_b \mathrm{e}^{\mathrm{sjs}(\beta_1,\beta_2,\ldots,\beta_k)}$, the sum ${}^k\mathring{a} + {}^k\mathring{b} = {}^k\mathring{s} = r_s \mathrm{e}^{\mathrm{sjs}(\vartheta_1,\vartheta_2,\ldots,\vartheta_k)}$. The equations (A.2) to (A.14) below show that for any ${}^k\mathring{a} + {}^k\mathring{b}$ there exists a r_s and $\mathrm{s}(\vartheta_1,\vartheta_2,\ldots,\vartheta_k)$ evaluated in terms of $r_a, r_b, \alpha_1, \alpha_2, \ldots, \alpha_k$ and $\beta_1, \beta_2, \ldots, \beta_k$.

$$k = 0:\ r_s = r_a + r_b \tag{A.1}$$

$$k = 1:\ \vartheta_1 = \arctan\left(\frac{r_a \sin\alpha_1 + r_b \sin\beta_1}{r_a \cos\alpha_1 + r_b \cos\beta_1}\right) \tag{A.2}$$

$$r_s = \sqrt{r_a^2 + r_b^2 + 2r_a r_b \cos(\alpha_1 - \beta_1)} \tag{A.3}$$

$$k = 2:\ \vartheta_1 = \arctan\left(\frac{r_a \sin\alpha_1 \cos\alpha_2 + r_b \sin\beta_1 \cos\beta_2}{r_a \cos\alpha_1 \cos\alpha_2 + r_b \cos\beta_1 \cos\beta_2}\right) \tag{A.4}$$

$$x_{2'} = \sqrt{r_a^2 \cos^2\alpha_2 + r_b^2 \cos^2\beta_2 + 2r_a r_b \cos(\alpha_1 - \beta_1)\cos\alpha_2 \cos\beta_2}$$

$$\vartheta_2 = \arctan\left(\frac{r_a \sin\alpha_2 + r_b \sin\beta_2}{x_{2'}}\right) \tag{A.5}$$

$$\begin{aligned} r_s &= \sqrt{x_{2'}^2 + (r_a \sin\alpha_2 + r_b \sin\beta_2)^2} \\ &= \sqrt{r_a^2 + r_b^2 + 2r_a r_b\big(\cos(\alpha_1 - \beta_1)\cos\alpha_2 \cos\beta_2 + \sin\alpha_2 \sin\beta_2\big)} \end{aligned} \tag{A.6}$$

$$k = 3:\ \vartheta_1 = \arctan\left(\frac{r_a \sin\alpha_1 \cos\alpha_2 \cos\alpha_3 + r_b \sin\beta_1 \cos\beta_2 \cos\beta_3}{r_a \cos\alpha_1 \cos\alpha_2 \cos\alpha_3 + r_b \cos\beta_1 \cos\beta_2 \cos\beta_3}\right) \tag{A.7}$$

$$\begin{aligned} x_{3'} = \Big[& r_a^2 \cos^2\alpha_2 \cos^2\alpha_3 + r_b^2 \cos^2\beta_2 \cos^2\beta_3 \\ & + 2r_a r_b \cos(\alpha_1 - \beta_1)\cos\alpha_2 \cos\beta_2 \cos\alpha_3 \cos\beta_3 \Big]^{1/2} \end{aligned}$$

$$\vartheta_2 = \arctan\left(\frac{r_a \sin\alpha_2 \cos\alpha_3 + r_b \sin\beta_2 \cos\beta_3}{x_{3'}}\right) \tag{A.8}$$

$$
\begin{aligned}
x_{3''} &= \sqrt{x_{3'}^2 + (r_a \sin\alpha_2 \cos\alpha_3 + r_b \sin\beta_2 \cos\beta_3)^2} \\
&= \Big[r_a^2 \cos^2\alpha_3 + r_b^2 \cos^2\beta_3 \\
&\qquad + 2 r_a r_b \Big(\cos\big(\alpha_1 - \beta_1\big) \cos\alpha_2 \cos\beta_2 + \sin\alpha_2 \sin\beta_2 \Big) \cos\alpha_3 \cos\beta_3 \Big]^{1/2} \\
\vartheta_3 &= \arctan\left(\frac{r_a \sin\alpha_3 + r_b \sin\beta_3}{x_{3''}} \right)
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
r_s &= \sqrt{x_{3''}^2 + (r_a \sin\alpha_3 + r_b \sin\beta_3)^2} \\
&= \Big[r_a^2 + r_b^2 + 2 r_a r_b \Big\{ \Big(\cos\big(\alpha_1 - \beta_1\big) \cos\alpha_2 \cos\beta_2 + \sin\alpha_2 \sin\beta_2 \Big) \\
&\qquad\qquad \times \cos\alpha_3 \cos\beta_3 \; + \; \sin\alpha_3 \sin\beta_3 \Big\} \Big]^{1/2}
\end{aligned} \tag{A.10}
$$

$$
k = n: \; \vartheta_1 = \arctan\left(\frac{r_a \sin\alpha_1 \cos\alpha_2 \cos\alpha_3 \ldots \cos\alpha_n + r_b \sin\beta_1 \cos\beta_2 \cos\beta_3 \ldots \cos\beta_n}{r_a \cos\alpha_1 \cos\alpha_2 \cos\alpha_3 \ldots \cos\alpha_n + r_b \cos\beta_1 \cos\beta_2 \cos\beta_3 \ldots \cos\beta_n} \right) \tag{A.11}
$$

$$
\begin{aligned}
x_{n'} &= \Big[r_a^2 \cos^2\alpha_2 \cos^2\alpha_3 \ldots \cos^2\alpha_n + r_b^2 \cos^2\beta_2 \cos^2\beta_3 \ldots \cos^2\beta_n \\
&\qquad + 2 r_a r_b \cos\big(\alpha_1 - \beta_1\big) \cos\alpha_2 \cos\beta_2 \cos\alpha_3 \cos\beta_3 \ldots \cos\alpha_n \cos\beta_n \Big]^{1/2} \\
\vartheta_2 &= \arctan\left(\frac{r_a \sin\alpha_2 \cos\alpha_3 \cos\alpha_4 \ldots \cos\alpha_n + r_b \sin\beta_2 \cos\beta_3 \cos\beta_4 \ldots \cos\beta_n}{x_{3'}} \right)
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
x_{n''} &= \sqrt{x_{3'}^2 + (r_a \sin\alpha_2 \cos\alpha_3 cos\alpha_4 \ldots \cos\alpha_n + r_b \sin\beta_2 \cos\beta_3 \cos\beta_4 \ldots \cos\beta_n)^2} \\
&\;\vdots \\
x_{n''\ldots''} &= \sqrt{x_{n''\ldots'}^2 + r_a^2 \sin^2\alpha_2 \cos^2\alpha_3 \ldots \cos^2\alpha_n + r_b^2 \sin^2\beta_2 \cos^2\beta_3 \ldots \cos^2\beta_n} \\
\vartheta_n &= \arctan\left(\frac{r_a \sin\alpha_n + r_b \sin\beta_n}{x_{3''\ldots''}} \right)
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
r_s &= \sqrt{x_{n''\ldots''}^2 + (r_a^2 \sin\alpha_n + r_b^2 \sin\beta_n})^2 \\
&= \Big[r_a^2 + r_b^2 + 2 r_a r_b \Big\{ \Big(\ldots \Big\{ \Big(\cos\big(\alpha_1 - \beta_1\big) \cos\alpha_2 \cos\beta_2 + \sin\alpha_2 \sin\beta_2 \Big) \\
&\qquad\qquad \times \cos\alpha_3 \cos\beta_3 \; + \; \sin\alpha_3 \sin\beta_3 \Big\} \ldots \Big) \\
&\qquad\qquad \times \cos\alpha_n \cos\beta_n \; + \; \sin\alpha_n \sin\beta_n \Big\} \Big]^{1/2}
\end{aligned} \tag{A.14}
$$

Appendix B; Non distributivity of multiplication over addition.

Spatial numbers of rank $k \geq 2$ are not distributive over multiplication.

$k = 0$: Distributivity holds; $r_a r_c + r_b r_c = r_c (r_a + r_b)$

$k = 1$: Distributivity holds; as identical to complex numbers and demonstrated by (A.2), which then is simplified by trigonometric reduction to obtain

$$\vartheta_1 = \arctan\left(\frac{r_a r_c \sin(\gamma_1+\alpha_1)+r_b r_c \sin(\gamma_1+\beta_1)}{r_a r_c \cos(\gamma_1+\alpha_1)+r_b r_c \cos(\gamma_1+\beta_1)}\right) = \gamma_1 + \arctan\left(\frac{r_a \sin\alpha_1 + r_b \sin\beta_1}{r_a \cos\alpha_1 + r_b \cos\beta_1}\right)$$

and from (A.3)

$$\sqrt{r_a^2 r_c^2 + r_b^2 r_c^2 + 2 r_a r_b r_c^2 \cos(\alpha_1+\gamma_1-(\beta_1+\gamma_1))} = r_c\sqrt{r_a^2 + r_b^2 + 2 r_a r_b \cos(\alpha_1-\beta_1)}$$

Therefor $r_a r_c \mathrm{e}^{\mathrm{sjs}(\alpha_1+\gamma_1)} + r_b r_c \mathrm{e}^{\mathrm{sjs}(\beta_1+\gamma_1)} = r_c \mathrm{e}^{\mathrm{sjs}(\gamma_1)} \left(r_a \mathrm{e}^{\mathrm{sjs}(\alpha_1)} + r_b \mathrm{e}^{\mathrm{sjs}(\beta_1)}\right)$

$k \geq 2$**:** Distributivity breaks down with $k \geq 2$; evident from (A.5), as

$$\vartheta_2 = \arctan\left(\frac{r_a r_c \sin(\gamma_2+\alpha_2)+r_b r_c \sin(\gamma_2+\beta_2)}{x_{2'}}\right) \neq \gamma_2 + \arctan\left(\frac{r_a \sin\alpha_2 + r_b \sin\beta_2}{x_{2'}/r_c}\right)$$

Therefor

$$r_a r_c \mathrm{e}^{\mathrm{sjs}(\alpha_1+\gamma_1,\alpha_2+\gamma_2)} + r_b r_c \mathrm{e}^{\mathrm{sjs}(\beta_1+\gamma_1,\beta_2+\gamma_2)} \neq r_c \mathrm{e}^{\mathrm{sjs}(\gamma_1,\gamma_2)}\left(r_a \mathrm{e}^{\mathrm{sjs}(\alpha_1,\alpha_2)} + r_b \mathrm{e}^{\mathrm{sjs}(\beta_1,\beta_2)}\right)$$