A mathematical derivation of the Maxwell equations in vacuum

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Slide 2: What is a wave? (The d'Alembert wave equation)

Towne[1] states that the requirement for a physical condition to be referred to as a wave, is that its mathematical representation give rise to a partial differential equation of particular form, known as the wave equation. The classical form

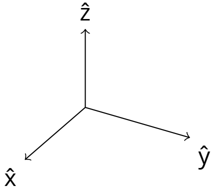
$$\frac{\partial^2 w}{\partial p^2} - \frac{1}{u^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{or} \qquad \nabla^2 w - \frac{1}{u^2}\frac{\partial^2 w}{\partial t^2} = 0.$$

was proposed in 1748 by d'Alembert for a one-dimensional continuum. A decade later, Euler established the equation for the three-dimensional continuum.

1 Dudley H. Towne. *Wave phenomena.* New York: Dover Publications, 1988.

Slide 3: A Cartesian reference system

The reference system whose axis are the unit vectors $\hat{x}, \hat{y}$ and $\hat{z}$



is defined by:

$$\hat{x} \cdot \hat{y} = 0 \qquad \hat{y} \cdot \hat{z} = 0 \qquad \hat{z} \cdot \hat{x} = 0$$

$$\hat{x} \times \hat{y} = \hat{z} \qquad \hat{y} \times \hat{z} = \hat{x} \qquad \hat{z} \times \hat{x} = \hat{y}$$

Slide 4: New: The Bimodal Wave Equation

Let's define three vectors:

$$\begin{aligned} \mathbf{u} &= c\,\hat{\mathrm{u}}(t) && \text{a velocity vector} \\ \mathbf{a} &= a\,\hat{\mathrm{a}}(t) && \text{an activator} \\ \mathbf{r} &= r\,\hat{\mathrm{r}}(t) && \text{and a reactivator} \end{aligned}$$

all are functions of time only. They are all orthogonal to each other, hence $\mathbf{u}\cdot\mathbf{a} = 0$, $\mathbf{a}\cdot\mathbf{r} = 0$ and $\mathbf{r}\cdot\mathbf{u} = 0$. But

$$\mathbf{r}_1 = \mathbf{u}_0 \times \mathbf{a}_0, \quad \mathbf{u}_1 = \mathbf{a}_0 \times \mathbf{r}_1, \quad \mathbf{a}_1 = \mathbf{r}_1 \times \mathbf{u}_1, \quad \mathbf{r}_2 = \mathbf{u}_1 \times \mathbf{a}_1, \quad \cdots$$

is an infinite series, or sequence.

Slide 5: New: The Bimodal Wave Equation

We require the sequence to continue unaltered indefinitely; that is $\mathbf{u}_n = \mathbf{u}$, $\mathbf{a}_n = \mathbf{a}$ and $\mathbf{r}_n = \mathbf{r}$, that needs some form of normalisation:

$$\left\{\mathbf{r} = \mathbf{u} \times \mathbf{a}, \quad \mathbf{u} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a} \times \mathbf{r}, \quad \mathbf{a} = \frac{1}{\|\mathbf{u}\|^2}\mathbf{r} \times \mathbf{u}\right\}$$

The solution of the above set of **three simultaneous vector equations** describes **bimodal-transverse waves**.

Slide 6: Electromagnetic Bimodal Wave Equation & Maxwell

Let's map $\mathbf{a} \mapsto \mathbf{B}$ and $\mathbf{r} \mapsto \mathbf{E}$ and we consider one plane $\mathcal{W}(\mathbf{p})$ of an EM-travelling plane wave and where $\mathbf{p}$ defines the position of $\mathcal{W}$. Such a wave is described (mathematically expressed: $\xrightarrow[\text{by}]{\text{dsc}}$) by the solution of $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$, which is a system of three simultaneous vector algebraic equations:

$$\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E}) = \left\{ \begin{array}{lr} \mathbf{E} = \mathbf{u} \times \mathbf{B} & (\textit{activation by } \mathbf{B}) \quad \text{(a)} \\ \mathbf{u} = \dfrac{1}{\|\mathbf{B}\|^2} \mathbf{B} \times \mathbf{E} & (\textit{vectoring by } \mathbf{B} \times \mathbf{E}) \quad \text{(b)} \\ \mathbf{B} = \dfrac{1}{\|\mathbf{u}\|^2} \mathbf{E} \times \mathbf{u} & (\textit{reactivation by } \mathbf{E}) \quad \text{(c)} \end{array} \right\}$$

Slide 7: Electromagnetic Bimodal Wave Equation & Maxwell

- $\mathbf{u}$ is a velocity vector $\mathbf{u} = c\hat{\mathrm{u}}$, where
- $\hat{\mathrm{u}}$ that is, $\hat{\mathrm{u}} \mapsto \hat{\mathrm{u}}(t)$, is a unitless unit vector function of time only, and
- c is the speed of light.
- $\mathbf{B}$ is the magnetic field $\mathbf{B} = B\hat{\mathrm{B}}$, and where
- $\hat{\mathrm{B}}$ that is, $\hat{\mathrm{B}} \mapsto \hat{\mathrm{B}}(t)$, is a unitless unit vector function of time only, and is orthogonal to $\hat{\mathrm{u}}$ hence $\hat{\mathrm{u}} \cdot \hat{\mathrm{B}} = 0$, and
- B scales the magnetic field and provides the physical units.
- $\mathbf{E}$ is the electric field and (a) gives $\mathbf{E} = cB(\hat{\mathrm{u}} \times \hat{\mathrm{B}}) = cB\hat{\mathrm{E}}$, with $\hat{\mathrm{E}} = \hat{\mathrm{u}} \times \hat{\mathrm{B}}$.
- $\mathbf{p}$ the position of the origin for $\mathbf{u}$, $\mathbf{B}$, and $\mathbf{E}$; thus $\mathbf{p} = \int \mathbf{u}\,\mathrm{d}t$.

Slide 8: Electromagnetic Bimodal Wave Equation & Maxwell

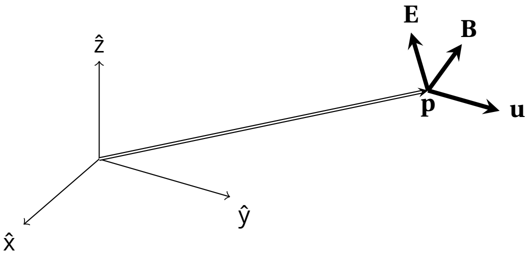


Figure 1: Illustrating the vectors used in $\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$.

Slide 9: Electromagnetic Bimodal Wave Equation & Maxwell

$$\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E}) = \left\{ \begin{array}{lr} \mathbf{E} = \mathbf{u}\times\mathbf{B} & (\textit{activation by}\ \mathbf{B})\ \ \text{(a)} \\[2ex] \mathbf{u} = \dfrac{1}{\|\mathbf{B}\|^2}\mathbf{B}\times\mathbf{E} & (\textit{vectoring by}\ \mathbf{B}\times\mathbf{E})\ \ \text{(b)} \\[2ex] \mathbf{B} = \dfrac{1}{\|\mathbf{u}\|^2}\mathbf{E}\times\mathbf{u} & (\textit{reactivation by}\ \mathbf{E})\ \ \text{(c)} \end{array} \right\}$$

$\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ predicts the Maxwell equations in vacuum, that is, $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ is the fundamental mathematical explanation for the electromagnetic wave phenomena.

Slide 10: Electromagnetic Bimodal Wave Equation & Maxwell

To show that $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$ is the superordinated mathematical formulation for the Maxwell equations is the task we tackle now:

But, first we need to evaluate the triple vector products $\nabla\times(\mathbf{u}\times\mathbf{B})$ and $\nabla\times(\mathbf{E}\times\mathbf{u})$, which we expand using general vector analytic methods.

$$\nabla\times(\mathbf{u}\times\mathbf{B}) = \mathbf{u}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{u}$$

$$\nabla\times(\mathbf{E}\times\mathbf{u}) = \mathbf{E}(\nabla\cdot\mathbf{u}) - \mathbf{u}(\nabla\cdot\mathbf{E}) - (\mathbf{E}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{E}$$

Slide 11: Electromagnetic Bimodal Wave Equation & Maxwell

$\nabla \cdot \mathbf{u} = 0$ because c and $\hat{\mathrm{u}}(t)$ are not functions of x, y, and z

$\nabla \cdot \mathbf{B} = 0$ because B and $\hat{\mathrm{B}}(t)$ are not functions of x, y, and z

$\nabla \cdot \mathbf{E} = 0$ ditto, because $\mathbf{E} = \mathbf{u} \times \mathbf{B}$

$(\mathbf{B} \cdot \nabla)\mathbf{u} = 0$ because $\left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right) c\hat{\mathrm{u}}(t) = 0$

$(\mathbf{E} \cdot \nabla)\mathbf{u} = 0$ ditto

$(\mathbf{u} \cdot \nabla)\mathbf{B} = ?$

$(\mathbf{u} \cdot \nabla)\mathbf{E} = ?$

Slide 12: Electromagnetic Bimodal Wave Equation & Maxwell

$$\mathbf{u}\cdot\nabla = \frac{\partial}{\partial t} \text{ because } \mathbf{u}\cdot\nabla = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial t}$$

and that leaves us with

$$\nabla\times(\mathbf{u}\times\mathbf{B}) = \cancel{\mathbf{u}(\nabla\cdot\mathbf{B})} - \cancel{\mathbf{B}(\nabla\cdot\mathbf{u})} + \cancel{(\mathbf{B}\cdot\nabla)\mathbf{u}} - (\mathbf{u}\cdot\nabla)\mathbf{B} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times(\mathbf{E}\times\mathbf{u}) = \cancel{\mathbf{E}(\nabla\cdot\mathbf{u})} - \cancel{\mathbf{u}(\nabla\cdot\mathbf{E})} + (\mathbf{u}\cdot\nabla)\mathbf{E} - \cancel{(\mathbf{E}\cdot\nabla)\mathbf{u}} = \frac{\partial\mathbf{E}}{\partial t}$$

Slide 13: Electromagnetic Bimodal Wave Equation & Maxwell

Applying a 'left and right side' curl operation on $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$(a) and (c) to obtain

$$\nabla\times\mathbf{E} = \nabla\times(\mathbf{u}\times\mathbf{B}) = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B} = \frac{1}{\|\mathbf{u}\|^2}\nabla\times(\mathbf{E}\times\mathbf{u}) = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{1}$$

and on slide 10 we established $\nabla\cdot\mathbf{B} = 0$ and $\nabla\cdot\mathbf{E} = 0$. Thus we have the Maxwell equations in vacuum if we can show that $c^{-2} = \epsilon_0\mu_0$.

Slide 14: Electromagnetic Bimodal Wave Equation & Maxwell

Because, $\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$ and $\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$

are derived from the Maxwell equations, proves that $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ is a new formulation for bimodal-waves as per Towne[2] (Slide 3)

$$\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E}) = \left\{ \begin{array}{lll} \mathbf{E} = \mathbf{u} \times \mathbf{B} & (\textit{activation by } \mathbf{B}) & \text{(a)} \\ \mathbf{u} = \dfrac{1}{\|\mathbf{B}\|^2} \mathbf{B} \times \mathbf{E} & (\textit{vectoring by } \mathbf{B} \times \mathbf{E}) & \text{(b)} \\ \mathbf{B} = \dfrac{1}{\|\mathbf{u}\|^2} \mathbf{E} \times \mathbf{u} & (\textit{reactivation by } \mathbf{E}) & \text{(c)} \end{array} \right\}$$

2 Towne, *Wave phenomena*.

Slide 15: Properties of Vacuum (Two Assertions)

To prove that $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ is also a reformulation of the Maxwell equations, we can take the easy path and simply substitute $c^2 = 1/\epsilon_0\mu_0$ in the above. The more difficult path is to assert

a) An elementary EM-wave $\mathcal{W}$ exhibits power h/t^2, where h is the Planck constant and $t = 1$ second. This requires $\mathbf{B}$ to be an elementary field.

b) This elementary wave transports an electric charge e every one second which is a *wave current.* (This is nothing new; it is another way of describing the displacement current $\partial\mathbf{E}/\partial t$ that Maxwell had identified in varying electric fields.)

Slide 16: Properties of Vacuum

From the definitions (Slide 7) we have $\|\mathbf{B}\| = B$ which we substitute into $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$(b) to obtain

$$\mathbf{u} = \frac{1}{B^2}\mathbf{B}\times\mathbf{E} \tag{2}$$

On the premise that $\mathbf{B}\times\mathbf{E}$ is indicative of the wave action, we multiply (2) by the quantised action h and evaluate the norms

$$\|h\mathbf{u}\| = \left\|\frac{h}{B^2}\mathbf{B}\times\mathbf{E}\right\| \qquad \text{to get} \qquad hc = \left(\frac{h}{B^2}\right)\left(|\mathbf{B}||\mathbf{E}|\right)$$

Slide 17: Properties of Vacuum

$$hc = \left(\frac{h}{B^2}\right)\left(|\mathbf{B}||\mathbf{E}|\right) \qquad \text{gives} \qquad h = \left[\frac{h}{l^4 B^2 c}\right] l^4 \left(|\mathbf{B}||\mathbf{E}|\right)$$

after defining an elementary distance $l = ct$, and multiplying and dividing the above-left by l^4 but that also requires **B** and **E** to be elementary fields. Here the square brackets indicate the development of a physical constant, which we want to determine by eliminating B and l or rather expressing them in terms of elementary action h and elementary charge e.

Slide 18: Elementary Electromagnetic Action (Definition)

Let's define the elementary electromagnetic action as:

$$h_e = \varrho h \qquad \text{where } \varrho = 1 \text{ C kg}^{-1}.$$

Action is momentum times distance, hence

$h_e = \varrho h = \kappa lec$ where l is the elementary distance, and κ is a dimensionless proportionality constant of unknown value scaling lec to the EM-action h_e. We can also postulate that the electromagnetic action is proportional to the product of **B** and the elementary volume which the wave occupies

$h_e = \varrho h = \chi l^3 |\mathbf{B}|$ where χ is a constant with units and scaling to be determined.

Slide 19: Properties of Vacuum

Therefore we have $h_e = \kappa lec = \chi l^3 |\mathbf{B}|$ which gives

$$|\mathbf{B}| = \frac{\kappa ec}{\chi l^2}$$

and we substitute $|\mathbf{B}|$ from the above into $\left[\frac{h}{l^4 B^2 c}\right] l^4 \Big(|\mathbf{B}||\mathbf{E}|\Big)$ gives

$$h = \left[\frac{h}{l^4 B^2 c}\right] l^4 \Big(\frac{\kappa ec}{\chi l^2} |\mathbf{E}|\Big) \qquad \text{but, } |\mathbf{E}| = cB \text{ which gives}$$

$$h = \left[\frac{h}{l^4 B^2 c}\right] \left[\frac{1}{\chi}\right] \kappa l^2 e c^2 B$$

Slide 20: Properties of Vacuum

Repeating last equation

$$h = \left[\frac{h}{l^4 B^2 c} \right] \left[\frac{1}{\chi} \right] \kappa l^2 e c^2 B$$

We are now in the position to define the expression for

$B = \dfrac{h}{\kappa l^2 e}$ but only if

$$1 = \left[\frac{h}{l^4 B^2 c} \right] \left[\frac{1}{\chi} \right] c^2$$

Slide 21: Properties of Vacuum

but only if

$$1 = \left[\frac{h}{l^4 B^2 c}\right]\left[\frac{1}{\chi}\right] c^2 \qquad \text{and replacing } B = \frac{h}{\kappa l^2 e} \text{ gives}$$

$$1 = \left[\frac{\kappa^2 e^2}{hc}\right]\left[\frac{1}{\chi}\right] c^2 \qquad \text{which requires } \frac{1}{\chi} = \frac{h}{\kappa^2 e^2 c}, \text{ hence}$$

$$1 = \left[\frac{\kappa^2 e^2}{hc}\right]\left[\frac{h}{\kappa^2 e^2 c}\right] c^2 \quad \text{all that remains is to set } \kappa^2 = \frac{1}{2\alpha}, \text{ thus}$$

$$1 = \left[\frac{e^2}{2\alpha hc}\right]\left[\frac{2\alpha h}{e^2 c}\right] c^2 = \epsilon_0 \mu_0 c^2$$

Slide 22: Properties of Vacuum

I now obtained the sought after result

$$\epsilon_0 = \frac{e^2}{2\alpha hc} \quad \text{and} \quad \mu_0 = \frac{2\alpha h}{e^2 c}$$

an expressions first formulated in 1916 by Sommerfeld[3] but in a way to define the fine structure constant α.

This concludes the proof that the equation set $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ is a mathematical reformulation of the Maxwell equations, because now we can replace $1/c^2$ in (1), Slide-13, with $\epsilon_0\mu_0$ as we have derived it independently.

3 A. Sommerfeld. “Zur Quantentheorie der Spektrallinien”. In: *Annalen der Physik* 356.17 (1916), pp. 1–94.

Slide 23: Discussion: Fine Structure Constant

The fine structure constant α is said to quantify the strength of the electromagnetic interaction between elementary charged particles; the modern view also includes the coupling of the electromagnetic force to the other three forces[4]; these are the strong, weak and gravitational forces. Repeating the equation for electromagnetic momentum, $\varrho h = \kappa lec$, here the constant $\kappa = 1/\sqrt{2\alpha}$ is a coupling constant relating the electric charge momentum to mechanical momentum.

4 *Current advances: The fine-structure constant and quantum Hall effect.* 2021. URL: `https://physics.nist.gov/cuu/Constants/alpha.html` (visited on 2021-10-30).

Slide 24: Discussion: Oliver Heaviside

Oliver Heaviside[5] in 1892 presented us with vector algebra,

Planck proposed the quantity h in the year 1900 and

Millikan published in 1913 the electric charge as 1.592×10^{-19} C.

What if Heaviside discovered $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$?

5 Oliver Heaviside. *Electromagnetic Theory.* Chelsea Publishing co., New York, 1893.

 Discussion: What if

A Heaviside constant $\kappa = 8.277$ would have been proposed,

The magnetic permeability could have transitioned from the fixed constant $4\pi \times 10^{-7}$ to the expression $h/(\kappa^2 e^2 c)$, or at least that relationship would have been known.

Then in 1916 Sommerfeld would have established $\alpha^{-1} = 2\kappa^2$.

Slide 26: Discussion: Elementary Length and time

Nonetheless, $\varrho h = \kappa lec$ now provides a key to determine the values for the elementary length and time. Using the 2018 CODATA values we get:

$$
\begin{aligned}
\kappa &= 8.277\,559\,999\,29(62) && \text{Heaviside constant} \\
l_0 &= 1.666\,566\,299\,11(12) \times 10^{-24} && \text{metres} \\
t_0 &= 5.559\,066\,796\,49(42) \times 10^{-33} && \text{seconds} \qquad \text{using } l = ct.
\end{aligned}
$$

Slide 27: Discussion: Magnetic flux quantum

The magnetic flux quantum is defined as

$$\phi_0 = h/(2e)$$

it was observed experimentally in 1961 by Deaver[6] in hollow superconducting cylinders, and Shankar[7] shows how to derive it by analysing the Aharonov–Bohm effect.

6 Bascom S. Deaver and William M. Fairbank. "Experimental Evidence for Quantized Flux in Superconducting Cylinders". In: *Physical Review Letters* 7.2 (1961-07), pp. 43–46. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.7.43` (visited on 2022-06-15).

7 R. Shankar. *Principles of Quantum Mechanics, 2nd edition.* SPlenum Press, 1994. 676 pp.

Slide 28: Discussion: Magnetic field quanta?

I found the magnetic field of an elementary EM-wave $\mathcal{W}$ as $B = h/(\kappa l^2 e)$ (Slide 20) which implies a magnetic flux for the elementary EM-wave

$$\tilde{\phi} = h/(\kappa e) = \sqrt{2\alpha}\, h/e$$

which is clearly smaller than the magnetic flux quantum established by measurement and quantum theory. I offer no opinion whether the irrational $\tilde{\phi} = \sqrt{2\alpha}\, h/e$ is a quantum or not, other than to comment that when scaled this way we have

$$\mathbf{S} = \mu_0^{-1} \|\mathbf{B} \times \mathbf{E}\| = h\frac{c^2}{l^4} \qquad \text{energy per (area} \times \text{time)}$$

which confirms the first of the assertions on Slide-15.

Slide 29: Discussion: What defines the speed of light?

‘*Who was first? The chicken or the egg*’

The values for the permittivity ϵ_0 and permeability μ_0 were derived using the velocity c defined previously in $\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E})$, and by the two assertions which introduced the unit action h and the elementary charge e. Therefore:

The relation

$$c = 1/\sqrt{\epsilon_0\mu_0}$$

does not define the speed of light from first principles!

Proposition: The vacuum has additional characteristics which defines the *transportivity* $\mathcal{T} = c^2$. As an analog to fluids, the transportivity could be a ratio of two properties, yet undiscovered, which are not functions of the speed of light.

For example, the speed of a sound wave in a material is dependent on the material properties. In fluids $c^2 = K_s/\rho$ where K_s is a coefficient of stiffness and ρ the fluid's density. Alternatively, we can also express it as $c^2 = \partial P/\partial\rho$ where P is pressure. But do take note of the fact that none of K_s, ρ and P are defined in terms of the speed of sound within that medium.

To fully describe a wave $\mathcal{W}$ also requires a set of parameter equations $\mathscr{P}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ which provide the solution to $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$. The parameter equations $\mathscr{P}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ define the unit vectors $\hat{\mathrm{u}}(t)$, $\hat{\mathrm{B}}(t)$ and $\hat{\mathrm{E}}(t)$ all as functions of t only, with the quantifiers c, B and E providing the necessary units, or quantities, and the scaling for $\mathcal{W}$. Any set of unit vectors $\hat{\mathrm{u}}(t)$, $\hat{\mathrm{B}}(t)$ and $\hat{\mathrm{E}}(t)$ that simultaneously satisfy

$$\hat{\mathrm{E}} = \hat{\mathrm{u}} \times \hat{\mathrm{B}} \qquad \hat{\mathrm{u}} = \hat{\mathrm{B}} \times \hat{\mathrm{E}} \qquad \hat{\mathrm{B}} = \hat{\mathrm{E}} \times \hat{\mathrm{u}}.$$

provide a solution to $\mathscr{M}$. Suitable solutions can be found, among others, by a succession of Euler rotations.

Slide 32: Describing EM-waves

$$\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{dsc}} \mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E}) = \left\{ \begin{aligned} &\mathbf{E} = \mathbf{u}\times\mathbf{B} \\ &\mathbf{u} = \frac{1}{\|\mathbf{B}\|^2}\mathbf{B}\times\mathbf{E} \\ &\mathbf{B} = \frac{1}{\|\mathbf{u}\|^2}\mathbf{E}\times\mathbf{u} \end{aligned} \right\} \xrightarrow[\text{by}]{\text{par}} \mathscr{P}(\mathbf{u},\mathbf{B},\mathbf{E})$$

which we simply shorten to:

$$\mathcal{W}(\mathbf{p}) \xrightarrow[\text{by}]{\text{par}} \mathscr{P}(\mathbf{u},\mathbf{B},\mathbf{E}) \qquad \text{where} \quad \mathbf{p} = \int \mathbf{u}\,\mathrm{d}t$$

Slide 33: Travelling plane waves

Every authoritative book, for example Jackson[8], describes an EM-field of a circular polarised travelling plane wave as

$$\begin{aligned}\mathbf{B}(\mathbf{z},t) &= B_0\big(\hat{\mathrm{x}}\cos(k\mathbf{n}\cdot\mathbf{z}-\omega t)+\hat{\mathrm{y}}\sin(k\mathbf{n}\cdot\mathbf{z}-\omega t)\big)\\ &= B_0\left[\hat{\mathrm{x}}\cos\left(\frac{\omega z}{c}-\omega t\right)+\hat{\mathrm{y}}\sin\left(\frac{\omega z}{c}-\omega t\right)\right]\end{aligned}$$

with the phase velocity defined by the wave vector $k\mathbf{n}$.

8 John David Jackson. *Classical Electrodynamics, 2nd. Ed.* John Wiley & Sons Inc, 1975.

Slide 34: Travelling plane waves

We define a wave's velocity vector $\mathbf{u} = \hat{z}c$ which gives the position vector $\mathbf{p}_i$ for the i[th] travelling plane

$$\mathbf{p}_i = \int \mathbf{u}\,\mathrm{d}t = \hat{z}\int c\,\mathrm{d}t = \hat{z}(z_i + ct)$$

and use it to describe a circular polarised travelling plane EM-wave $\mathcal{W}$ as $\mathcal{W}(\mathbf{p}_i, t) \xrightarrow[\text{by}]{\text{par}} \mathcal{P}_i(\mathbf{u}, \mathbf{B}, \mathbf{E}, t)$

$$\mathcal{P}_i(\mathbf{u}, \mathbf{B}, \mathbf{E}, t) = \left\{ \begin{aligned} &\mathbf{u} = \hat{z}c \\ &\mathbf{B} = B\left[\hat{x}\cos\omega\left(\frac{\mathbf{p}_i}{c} - t\right) + \hat{y}\sin\omega\left(\frac{\mathbf{p}_i}{c} - t\right)\right] \\ &\mathbf{E} = cB\left[-\hat{x}\sin\omega\left(\frac{\mathbf{p}_i}{c} - t\right) + \hat{y}\cos\omega\left(\frac{\mathbf{p}_i}{c} - t\right)\right] \end{aligned} \right\}$$

$\mathcal{W}(\mathbf{p}_i, t) \xrightarrow[\text{by}]{\text{par}} \mathscr{P}_i(\mathbf{u}, \mathbf{B}, \mathbf{E}, t)$ describes particular travelling plane of the wave $\mathcal{W}$ evaluated at the position $\mathbf{p}_i$ for any initial position $-\infty < z_i < \infty$ at $t = 0$. But why so complicated? With $\mathbf{p}_i = \hat{\mathsf{z}} \int c\,\mathrm{d}t = \hat{\mathsf{z}}(z_i + ct)$ we can simplify the above to

$$\mathcal{W}(\mathbf{p}_i) \xrightarrow[\text{by}]{\text{par}} \left\{ \begin{array}{l} \mathbf{u} = \hat{\mathsf{z}} c \\ \mathbf{B} = B\left[\hat{\mathsf{x}} \cos(\omega z_i / c) + \hat{\mathsf{y}} \sin(\omega z_i / c)\right] \\ \mathbf{E} = cB\left[-\hat{\mathsf{x}} \sin(\omega z_i / c) + \hat{\mathsf{y}} \cos(\omega z_i / c)\right] \end{array} \right\} \tag{3}$$

and we know that the above parameters provide a solution to $\mathscr{M}(\mathbf{u}, \mathbf{B}, \mathbf{E})$ and I have shown that $\mathscr{M}$ is a reformulation of the Maxwell equation in vacuum. Therefore (3) describes a particular plane of an EM-travelling plane wave.

Slide 36: Ball lightning as a three dimensional EM-soliton

For a travelling object $\mathcal{o}$ we define a unit velocity vector as

$$\hat{\mathrm{u}}_{\mathcal{o}}(t) = \hat{\mathrm{x}}\sin\omega_1 t\sin\omega_2 t + \hat{\mathrm{y}}\sin\omega_1 t\cos\omega_2 t + \hat{\mathrm{z}}\cos\omega_1 t$$

The path $\mathbf{s}_{\mathcal{o}}$ that the object $\mathcal{o}$ follows is found by integration

$$\begin{aligned}\mathbf{s}_{\mathcal{o}}(t) &= \int c\hat{\mathrm{u}}_{\mathcal{o}}\mathrm{d}t \\ &= \hat{\mathrm{x}}c\left(\frac{\sin(\omega_2+\omega_1)t}{2(\omega_2+\omega_1)} - \frac{\sin(\omega_1-\omega_2)t}{2(\omega_1-\omega_2)}\right) \\ &\quad + \hat{\mathrm{y}}c\left(\frac{\cos(\omega_2+\omega_1)t}{2(\omega_2+\omega_1)} + \frac{\cos(\omega_1-\omega_2)t}{2(\omega_1-\omega_2)}\right) - \hat{\mathrm{z}}c\frac{\sin\omega_1 t}{\omega_1}\end{aligned}$$

The path is closed, or repeats, in periods of $t = 2\pi$ and as $\|c\hat{\mathrm{u}}_{\mathcal{o}}(t)\| = c$ the pathlength of $\mathbf{s}_{\mathcal{o}}$ is $2\pi c$.

Slide 37: Ball lightning as a three dimensional EM-soliton

Let's define an EM-soliton Θ as a EM-wave that exists only on the closed path $\mathbf{p} = \int c\hat{\mathrm{u}}_{\mathcal{O}}\mathrm{d}t$, and at $t = 0$ it is at the position $\mathbf{p}_0 = \mathbf{s}_{\mathcal{O}}(t_0)$, thus $\mathbf{u} = c\,\hat{\mathrm{u}}_{\mathcal{O}}(t_0 + t)$, hence:

$$\Theta(\mathbf{p}, t) \xrightarrow[\text{by}]{\text{dsc}} \left\{ \begin{aligned} \mathbf{u} &= c\big(\hat{\mathrm{x}}\sin\omega_1(t_0+t)\sin\omega_2(t_0+t) \\ &\qquad +\hat{\mathrm{y}}\sin n\omega_1(t_0+t)\cos\omega_2(t_0+t) + \hat{\mathrm{z}}\cos\omega_1(t_0+t)\big) \\ \mathbf{B} &= B\big(\hat{\mathrm{x}}\cos\omega_2(t_0+t) - \hat{\mathrm{y}}\sin\omega_2(t_0+t)\big) \\ \mathbf{E} &= cB\big(\hat{\mathrm{x}}\cos\omega_1(t_0+t)\sin\omega_2(t_0+t) \\ &\qquad +\hat{\mathrm{y}}\cos\omega_1(t_0+t)\cos\omega_2(t_0+t) - \hat{\mathrm{z}}\sin\omega_1(t_0+t)\big) \end{aligned} \right\}$$

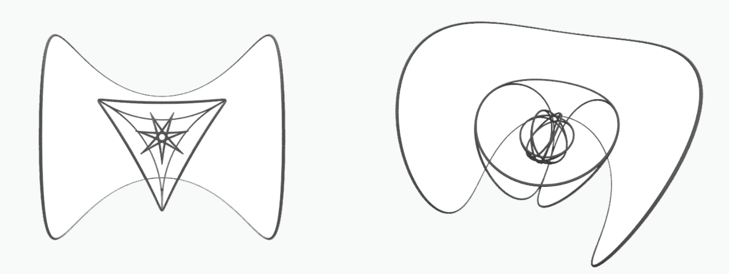
Slide 38: Ball lightning as a three dimensional EM-soliton

The above satisfies the equation set

$$\mathscr{M}(\mathbf{u},\mathbf{B},\mathbf{E}) = \left\{ \mathbf{E} = \mathbf{u}\times\mathbf{B}, \quad \mathbf{u}\frac{1}{\|\mathbf{B}\|^2}\mathbf{B}\times\mathbf{E}, \quad \mathbf{B} = \frac{1}{\|\mathbf{u}\|^2}\mathbf{E}\times\mathbf{u} \right\}$$

Therefore, $\Theta(\mathbf{p}, t)$ describes an EM-wave; propagating on the closed path $\mathbf{p} = \mathbf{s}_{\mathcal{O}}$ at velocity c. The equation set $\Theta(\mathbf{p}, t)$ suggests that the wave is "trapped" by its magnetic field which forms a closed ring that precesses with the wave motion, that is when connected to the magnetic field of a point retarded by $\omega_2(t_0 - ½)$ but at a different z-elevation; and with the electric field always radiating outwards.

Slide 39: Soliton



Two views of the path $\hat{s}_o(t)$ defined by $\Theta(\mathbf{p}, t)$ for the frequency ratios $\omega_1 : \omega_2 = 1:2$, $1:3$, and $1:7$. The path length of each curve is 2π.

Slide 40: Ball lightning as a three dimensional EM-soliton

Here I need to point out that Arnhoff[9], Chubykalo[10] and Cameron[11] presented solutions for three dimensional EM-wave structures, all of which are based on the superposition principle which allows the construction of intricate wave structures, and the complexity of all contrasts with the simplicity of $\Theta(\mathbf{p}, t)$.

9 G. Arnhoff. "Is there yet an explanation of ball lightning?" en. In: *European Transactions on Electrical Power* 2.3 (1992), pp. 137–142. URL: `https://onlinelibrary.wiley.com/doi/abs/10.1002/etep.4450020302` (visited on 2022-06-13).

10 Andrew E. Chubykalo and Augusto Espinoza. "Unusual formations of the free electromagnetic field in vacuum". en. In: *Journal of Physics A: Mathematical and General* 35.38 (2002-09), pp. 8043–8053. URL: `https://doi.org/10.1088/0305-4470/35/38/307` (visited on 2022-06-10).

11 Robert P Cameron. "Monochromatic knots and other unusual electromagnetic disturbances: light localised in 3D". in: *Journal of Physics Communications* 2.1 (2018), p. 015024.

Slide 41: Soliton

The term *soliton* describes self reinforcing solitary waves. Drazin[12] defined a soliton as any solution of a nonlinear equation (or a system) which:

i. *represents a wave of permanent form;*

ii. *is localised, so that it decays or approaches a constant at infinity;*

iii. *can interact strongly with other solitons and retain its identity.*

The above analysis confirms the first two points; the third is yet to be demonstrated.

12 P. G. Drazin and R. S. Johnson. *Solitons: An Introduction.* Cambridge University Press, 2002.

Slide 42: Conclusion

With this talk I presented a novel wave equation system, where each possible solution describes a bimodal-transverse wave in a more *aufschlussreicher*[13] way than was possible with the classic partial differential approach. Remarkably, this equation set proves to be fundamental to electromagnetic theory as it demands the formulations for ϵ_0 and μ_0 in a form previously only derivable from seemingly unrelated atomic theories and physical observations.

13 aufschlussreich: German adj., translations: enlightening, illuminating, informative, insightful, instructive, revealing, and telling.